

Emergent Cosmology Revisited

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Abstract. We explore the possibility of emergent cosmology using the *effective potential* formalism. We discover new models of emergent cosmology which satisfy the constraints posed by the cosmic microwave background (CMB). We demonstrate that, within the framework of modified gravity, the emergent scenario can arise in a universe which is spatially open/closed. By contrast, in general relativity (GR) emergent cosmology arises from a spatially closed past-eternal Einstein Static Universe (ESU). In GR the ESU is unstable, which creates fine tuning problems for emergent cosmology. However, modified gravity models including Braneworld models, Loop Quantum Cosmology (LQC) and Asymptotically Free Gravity result in a stable ESU. Consequently, in these models emergent cosmology arises from a larger class of initial conditions including those in which the universe eternally oscillates about the ESU fixed point. We demonstrate that such an oscillating universe is necessarily accompanied by *graviton production*. For a large region in parameter space graviton production is enhanced through a parametric resonance, casting serious doubts as to whether this emergent scenario can be past-eternal.

Keywords: Effective potential, Einstein Static Universe, Inflation, Emergent Cosmology, CMB constraints on inflation, Particle production

ArXiv ePrint: [1403.4243](https://arxiv.org/abs/1403.4243)

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1 Introduction

The inflationary scenario has proved to be successful in describing a universe which is remarkably similar to the one which we inhabit. Indeed, one of the central aims of the ongoing effort in the study of cosmic microwave background (CMB) observations is to converge on the correct model describing inflation [1].

However, despite its very impressive achievements, the inflationary paradigm leaves some questions unanswered. These pertain both to the nature of the inflaton field and to the state of the universe prior to the commencement of inflation. Indeed, as originally pointed out in [2], inflation (within a general relativistic setting) could not have been past eternal. This might be seen to imply one of several alternative possibilities including the following:

1. The universe quantum mechanically tunnelled into an inflationary phase.

2. The universe was dominated by radiation (or some other form of matter) prior to inflation and might therefore have encountered a singularity in its past.
3. The universe underwent a non-singular bounce prior to inflation. Before the bounce the universe was contracting.
4. The universe existed ‘eternally’ in a quasi-static state, out of which inflationary expansion emerged.

One should point out that, at the time of writing, none of the above possibilities is entirely problem free. Nevertheless, our focus in this paper will be on the last option, namely that of an *Emergent Cosmology*.

The idea of an emergent universe is not new and an early semblance of this concept can be traced back to the seminal work of Eddington [3] and Lemaître [4], which was based on the Einstein Static Universe [5]. Indeed, in 1917, Einstein introduced the idea of a closed and static universe sourced by a cosmological constant and matter. Subsequently it was found that: (a) the observed universe was expanding [6], (b) the Einstein Static Universe (ESU) was unstable. It therefore became unlikely that ESU could describe the present universe but allowed for our universe to have emerged from a static ESU-phase in the past.

With the discovery of cosmic expansion Einstein distanced himself from his own early ideas referring to them, years later, as his biggest blunder [7].

Interest in the ESU subsequently waned, although models in which the ESU featured as an intermediate stage — called *loitering* — received a short-lived burst of attention in the late 1960’s, when it was felt that a universe which loitered at $z \simeq 2$ might account for the abundance of QSO’s at that redshift; an observation that inspired Zeldovich to write his famous review on the cosmological constant [8].

The present resurgence of interest in ESU and emergent cosmology owes much to the CMB observations favouring an early inflationary stage, supplemented by the fact that an inflationary universe is geodesically incomplete [2] and might therefore have had a beginning.

This paper commences with a discussion of emergent cosmology in the context of general relativity in section 2. Since GR-based ESU is unstable, this scenario suffers from severe fine-tuning problems, as originally pointed out in [9, 10]. One can construct stable ESUs in the context of the Braneworld scenario [11], Loop Quantum Cosmology (LQC) [12] and Asymptotically Free Gravity. This is the focus of section 3. When viewed in the classical context, a stable ESU allows the universe to oscillate ‘eternally’ about the ESU fixed point [13, 14]. If the universe is filled with a scalar field, then these oscillations can end, giving rise to inflation. However, this scenario is feasible only for an appropriate choice of the inflaton potential. Equally important is the fact that an oscillating universe generically gives rise to graviton production, which forms the focus of section 4. For a large region in parameter space, the production of gravitons proceeds through a parametric resonance, which seems to question the possibility of whether a universe could have oscillated ‘eternally’ about the ESU fixed point. Our conclusions are drawn in section 5.

Our main results seem to suggest that, while emergent cosmology (EC) can be constructed on the basis of both GR and modified gravity, the restrictions faced by working EC models are many. Consequently, realistic EC is possible to construct only in a small region of parameter space, and that too for a rather restrictive class of inflationary potentials.

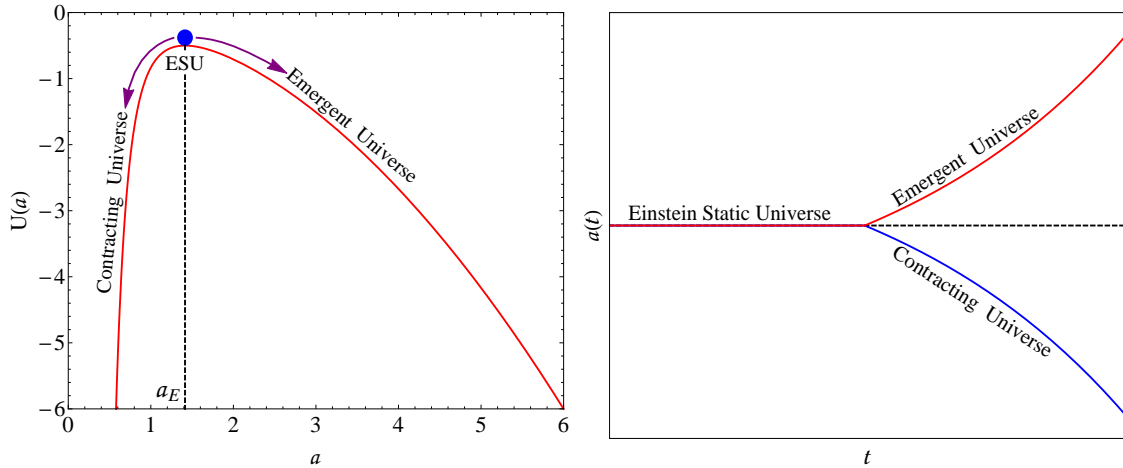


Figure 1. The effective potential (left panel) is schematically shown for a universe consisting of two components, one of which satisfies the strong energy condition $\rho + 3P \geq 0$, while the other violates it. The maxima of the effective potential corresponds to the Einstein Static Universe (ESU) for which the expansion factor is a constant. However ESU is unstable to small perturbations and can therefore be perturbed either into an accelerating emergent cosmology, or into a contracting singular universe (right panel).

2 Emergent cosmology and the effective potential formalism

An Einstein Static Universe (ESU) is possible to construct provided the universe is closed and is filled with at least two components of matter: one of which satisfies the strong energy condition (SEC) $\rho + 3P \geq 0$, whereas the other violates it. The cosmological constant with $P = -\rho$ presents us with an example of the latter, as does a massive scalar field which couples minimally to gravity.

In order to appreciate the existence of ESU, and therefore of emergent cosmology (EC), consider the following set of equations which describe the dynamics of a FRW universe:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3} \sum_{i=1}^2 \rho_i - \frac{\mathbb{k}}{a^2}, \quad \mathbb{k} = 0, \pm 1 \quad (2.1)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{\kappa}{6} \sum_{i=1}^2 (\rho_i + 3P_i), \quad (2.2)$$

where $\kappa = 8\pi G = M_p^{-2}$. We shall assume that the pressure of the first component satisfies the SEC ($w_1 \geq -1/3$), whereas that of the second component violates it ($w_2 < -1/3$), where $w = P/\rho$ is the parameter of equation of state.

Equation (2.1) can be recast in terms of the effective potential $U(a)$ as follows [15]:

$$\frac{1}{2}\dot{a}^2 + U(a) = E \equiv -\frac{\mathbb{k}}{2}, \quad (2.3)$$

where

$$U(a) = -\frac{\kappa}{6} a^2 \sum_{i=1}^2 \rho_i. \quad (2.4)$$

The ESU arises when the following two conditions are simultaneously satisfied:

$$\dot{a} = 0 \ , \quad \ddot{a} = 0 \ . \quad (2.5)$$

Substituting $\dot{a} = 0$ into (2.3) gives

$$U(a_E) = -\frac{\mathbb{k}}{2} \ , \quad (2.6)$$

where a_E is the scale factor for the ESU. The second condition implies that the ESU corresponds to an extremum of $U(a)$ (see FIG. 1):

$$\ddot{a} = 0 \ \Rightarrow \ U'(a_E) = 0 \ . \quad (2.7)$$

In this case, one finds the following relationship between the curvature of the ESU, a_E^{-2} , and the densities ρ_i for a closed universe ($\mathbb{k} = 1$):

$$\frac{\rho_1}{\rho_2} = -\frac{1+3w_2}{1+3w_1} \ , \quad \kappa\rho_2 = \left(\frac{1+3w_1}{w_1-w_2} \right) \frac{1}{a_E^2} \ . \quad (2.8)$$

For a closed Λ CDM ESU, we have $\rho_1 \equiv \rho_m$, $\kappa\rho_2 = \Lambda$, $w_1 = 0$, $w_2 = -1$, and equation (2.8) yields the familiar results

$$\Lambda = \frac{\kappa}{2}\rho_m = \frac{1}{a_E^2} \ . \quad (2.9)$$

On the other hand, for a closed *radiation* + Λ ESU, we have $\rho_1 \equiv \rho_r$, $\kappa\rho_2 = \Lambda$, $w_1 = 1/3$, $w_2 = -1$, and one finds

$$\Lambda = \kappa\rho_r = \frac{3}{2a_E^2} \ . \quad (2.10)$$

The form of $U(a)$ in Fig. 1 immediately suggests that the ESU with $U'(a_E) = 0$, $U''(a_E) < 0$ is unstable, and infinitesimally small homogeneous perturbations will either (i) cause the ESU to contract towards a singularity, or (ii) lead to an accelerating expansion at late times, in other words, to *Emergent Cosmology*. In this latter case, an exact solution describing the emergence of a Λ -dominated accelerating universe from a *radiation* + Λ based ESU is given by [16]

$$a(t) = a_i \left[1 + \exp \left(\frac{\sqrt{2}t}{a_i} \right) \right]^{1/2} \ , \quad (2.11)$$

where $a_i^2 = 3/(2\Lambda) = 3/(2\kappa\rho_r)$.

The above example provided us with a toy model for the emergent scenario in which ‘inflation’ takes place eternally. A more realistic scenario can be constructed if one replaces Λ by the inflaton [9] thereby allowing inflation to end and the universe to reheat.

2.1 Inflationary Emergent Cosmology

A spatially homogeneous scalar field minimally coupled to gravity is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \ , \quad (2.12)$$

and satisfies the evolution equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \ . \quad (2.13)$$

The energy density and pressure of such a field are given, respectively, by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) , \quad (2.14a)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) , \quad (2.14b)$$

and, by virtue of (2.13), satisfy the usual energy conservation equation

$$\dot{\rho}_\phi = -3H(\rho_\phi + P_\phi) . \quad (2.15)$$

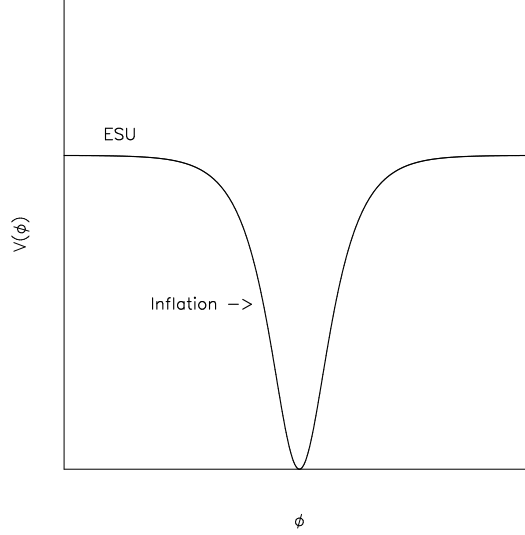


Figure 2. The potential (2.16) can give rise to an emergent cosmology commencing in a *Einstein Static Universe* (ESU). ESU is followed by inflation after which ϕ oscillates and the universe reheats. Since the potential is symmetric, emergent cosmology can be realized by the inflaton rolling either towards the right, or the left.

An emergent universe can be constructed if the inflaton potential $V(\phi)$ has one (or more) flat wings.

- Consider first the potential

$$V(\phi) = V_0 \tanh^{2p}(\lambda\phi/M_p) ; \quad p = 1, 2, 3, \dots \quad (2.16)$$

Large absolute values of $|\lambda\phi| \gg M_p$ lead to a flat potential with $V(\phi) \simeq V_0$; see Fig. 2. In this case, the equation of motion (2.13) becomes $\ddot{\phi} + 3H\dot{\phi} \simeq 0$. Since $H = 0$ in an ESU, one immediately finds $\dot{\phi}^2 = \text{const}$. Consequently, a scalar field sustaining an ESU behaves exactly like a two-component fluid, with one component being stiff matter with equation of state $P = \rho = \dot{\phi}^2/2$, while the other is the cosmological constant $\Lambda \equiv \kappa V_0$. Substituting $\rho_1 = \dot{\phi}^2/2$, $\rho_2 = V_0$, $w_1 = 1$, $w_2 = -1$ into (2.8), one gets

$$\dot{\phi}^2 = V_0 = \frac{2}{\kappa a_E^2} , \quad (2.17)$$

which demonstrates that the kinetic term must be *precisely matched* to the asymptotic value of the potential term in an ESU scenario; see also [9]. Note that the post-ESU inflationary phase can last for a sufficiently long duration, as illustrated in Fig. 3.

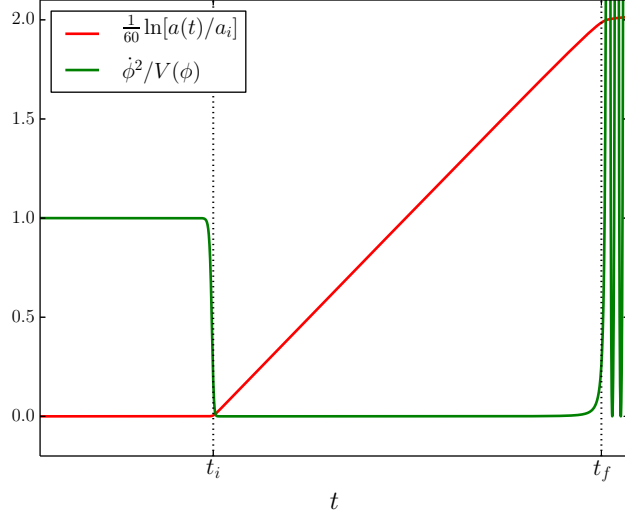


Figure 3. The emergent scenario is illustrated by the scale factor (red curve) and slow roll parameter (green curve). Inflation commences when $t \sim t_i$ and lasts for 120 e-folds. Note that the (quasi-flat) left branch of the potential in (2.16) needs to have a small positive slope in order for the ESU to end in inflation. If this branch has a negative slope, then the ESU will end in a contracting universe – see Fig. 1.

2.2 CMB Constraints

Next, we shall demonstrate that emergent cosmology based on (2.16) is consistent with recent measurements of the Cosmic Microwave Background (CMB) made by the Planck satellite [1]; see also [17].

As shown in Fig. 2, ESU ends and inflation commences once the inflaton field ϕ begins to roll down its potential. In the slow-roll approximation, the scalar (n_s) and tensor (n_T) spectral indices and the tensor-to-scalar ratio (r) are given by

$$n_s - 1 = - \frac{16 p \lambda^2 \left(p + 8 N p \lambda^2 + \sqrt{1 + 8 p^2 \lambda^2} \right)}{\left(8 N p \lambda^2 + \sqrt{1 + 8 p^2 \lambda^2} \right)^2 - 1}, \quad (2.18)$$

$$n_T = - \frac{16 p^2 \lambda^2}{\left(8 N p \lambda^2 + \sqrt{1 + 8 p^2 \lambda^2} \right)^2 - 1}, \quad (2.19)$$

$$r = \frac{128 p^2 \lambda^2}{\left(8 N p \lambda^2 + \sqrt{1 + 8 p^2 \lambda^2} \right)^2 - 1}, \quad (2.20)$$

respectively, where N is the number of e -folds counted from the end of inflation. The values of n_s and r are plotted as a function of λ in Fig. 4. We find that CMB constraints are easily satisfied if $\lambda > 0.1$.

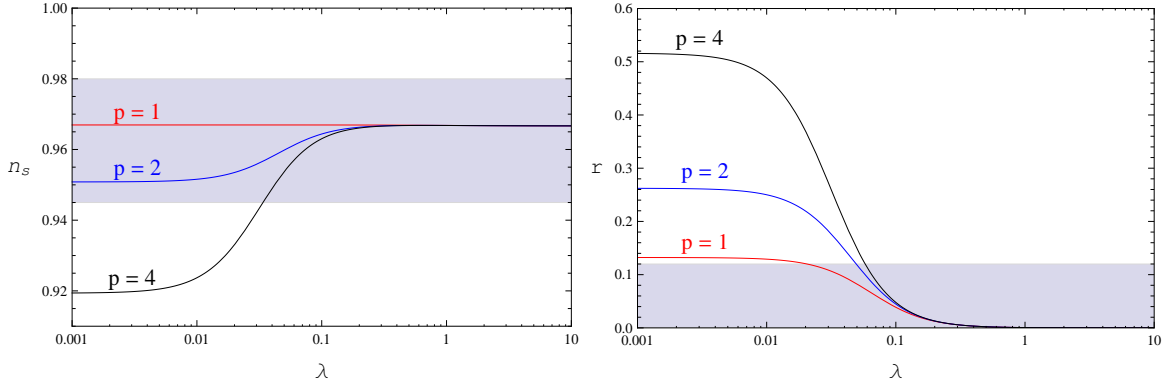


Figure 4. In the left panel, the scalar spectral index n_s is plotted as a function of λ for three different values of the parameter p in the potential (2.16) while, in the right panel, the value of tensor-to-scalar is plotted as a function of λ for the same set of values for p . The number of e -folds is $N = 60$. Note that when $\lambda > 0.1$, the scalar spectral index approaches a constant value of 0.967, whereas r decreases as λ^{-2} . The shaded region refers to 95% confidence limits on n_s and r determined by Planck [1].

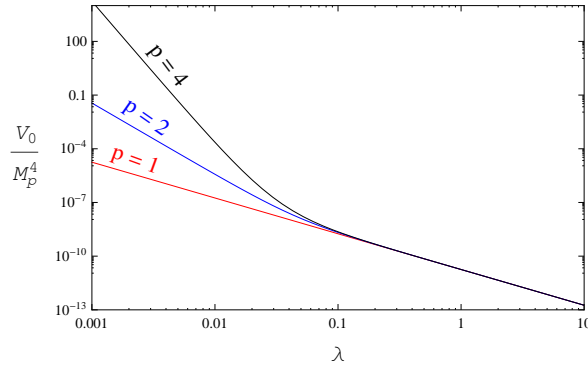


Figure 5. The value of V_0 is plotted as a function of λ for three different values of the parameter p in the potential (2.16). The number of e -folds is taken to be $N = 60$.

In this case, equations (2.18)–(2.20) are simplified, respectively, to

$$n_s - 1 \simeq -\frac{2}{N} , \quad (2.21)$$

$$n_T \simeq -\frac{1}{4N^2 \lambda^2} , \quad (2.22)$$

$$r \simeq \frac{2}{N^2 \lambda^2} , \quad (2.23)$$

which are *independent* of the value of p in (2.16).

Interestingly, the above expression for n_s is *exactly the same* as in Starobinsky's $R + R^2$ model [18]. In fact, for $\lambda = 1/\sqrt{6}$, the expressions for n_T and r match those in the Starobinsky inflation !

The value of the parameter V_0 in the potential (2.16) can be fixed using CMB normalization, which gives $P_s(k_*) = 2.2 \times 10^{-9}$ at the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$ [1]. One finds

$$\frac{V_0}{M_p^4} = \frac{192\pi^2 p^2 \lambda^2 \left(8Np\lambda^2 + \sqrt{1 + 8p^2\lambda^2} + 1\right)^{2p} P_s(k_*)}{\left[\left(8Np\lambda^2 + \sqrt{1 + 8p^2\lambda^2}\right)^2 - 1\right]^{p+1}}, \quad (2.24)$$

which is plotted as a function of λ in Fig. 5. Since $\lambda > 0.1$ is preferred from the CMB bounds on n_s and r , it follows that $V_0 < 10^{-8} M_p^4$.

- Consider next the potential

$$V(\phi) = V_0 \left[\exp\left(\frac{\beta\phi}{M_p}\right) - 1 \right]^2. \quad (2.25)$$

shown in Fig. 6 and earlier discussed in [10].

For suitable values of V_0 and β , inflation can occur both from the (flat) left branch and from the (steep) right branch of this potential.

In this section, we shall focus on the left branch since this is the branch which leads to an ESU, and hence to emergent cosmology in the GR context. For $\beta > 0.1$, one finds

$$n_s - 1 \simeq -\frac{2}{N} - \frac{3}{\beta^2 N^2}, \quad (2.26)$$

$$n_T \simeq -\frac{1}{\beta^2 N^2}, \quad (2.27)$$

$$r \simeq \frac{8}{\beta^2 N^2}. \quad (2.28)$$

If $N \geq 60$, then $\beta > 0.14$ is required in order to satisfy the CMB bound $r < 0.11$. The value of V_0 is determined from the relation

$$\frac{V_0}{M_p^4} = 12\pi^2 \left(\frac{P_s(k_*)}{N^2 \beta^2} \right). \quad (2.29)$$

Substituting here $\beta = 0.5$ gives $V_0 = 2.9 \times 10^{-10} M_p^4$, $n_s \simeq 0.96$ and $r \simeq 0.009$, which are in good agreement with the *Planck* results [1]. Note that Starobinsky's $R + R^2$ model of inflation [18] can equivalently be described by a scalar field with potential (2.25) but with $\beta = \sqrt{2/3}$; see [19].

To summarize, we have demonstrated that viable models of emergent cosmology are possible to construct in a general relativistic setting. Unfortunately, although our models easily satisfy CMB constraints, they run into severe fine-tuning issues — since the initial value of ϕ_i^2 must precisely match the asymptotic form of the inflaton potential — a consequence of the ESU being unstable in GR; see (2.17). (Note that emergent cosmology can also be sustained by a spinor field, as demonstrated in [20].)

As we shall show in the next section, emergent cosmology based on modified gravity can circumvent this difficulty since an ESU can be associated with a *stable* fixed point in this case.

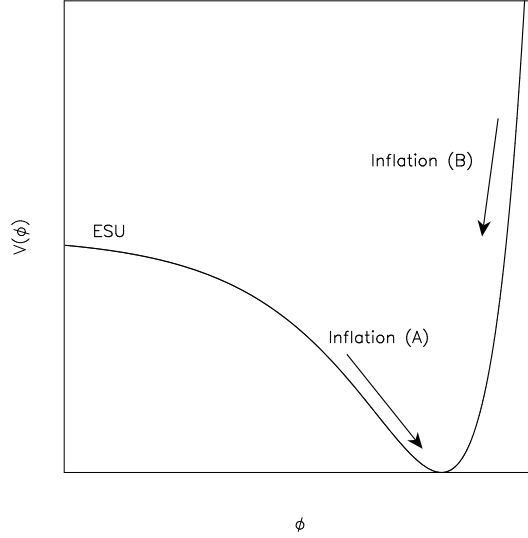


Figure 6. Potentials (2.25) and (3.33) can give rise to an emergent cosmology from a *Einstein Static Universe* (ESU). If the ESU is GR-based, as in section 2, then inflation will occur from the left (A) branch. For modified gravity models explored in section 3, inflation can occur from both A and B branches.

3 Emergent cosmology from modified gravity

In this section we focus on emergent cosmology based on modified gravity. We shall confine our discussion to three distinct models of modified gravity, namely: (i) Braneworld cosmology and its generalizations, (ii) Loop Quantum Cosmology (LQC), and (iii) a phenomenological model of *asymptotically free* gravity.

3.1 Emergent scenario in Braneworld Cosmology

Consider first the following generalization of the Einstein equations which is known to give rise to a non-singular bouncing cosmology [21]:

$$H^2 = \frac{\kappa}{3}\rho \left\{ 1 - \left(\frac{\rho}{\rho_c} \right)^m \right\} - \frac{\mathbb{k}}{a^2}, \quad \kappa = 8\pi G = M_p^{-2}, \quad (3.1)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6} \left[(\rho + 3P) - \left\{ (3m+1)\rho + 3(m+1)P \right\} \left(\frac{\rho}{\rho_c} \right)^m \right]. \quad (3.2)$$

The bounce occurs near the critical density ρ_c . When $m = 1$, these equations reduce to the following set of equations describing the dynamics of a Friedmann–Robertson–Walker (FRW) metric on a brane [11]

$$H^2 = \frac{\kappa}{3}\rho \left\{ 1 - \frac{\rho}{\rho_c} \right\} - \frac{\mathbb{k}}{a^2}, \quad (3.3)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6} \left\{ (\rho + 3P) - \frac{2\rho}{\rho_c}(2\rho + 3P) \right\}, \quad (3.4)$$

where ρ_c is related to the five-dimensional Planck mass [11]. Both (3.1), (3.2) & (3.3), (3.4) reduce to the FRW limit when $\rho_c \rightarrow \infty$, while (3.3), (3.4) are valid in LQC when $\mathbb{k} = 0$ [12].

In the presence of several components one replaces $\rho = \sum_i \rho_i$ and $P = \sum_i P_i$ in all these equations.

The emergent scenario which we discuss in this paper is based on ‘normal’ matter that satisfies the strong energy condition, along with a scalar field that violates it. As pointed out in the previous section, emergent cosmology arises when the scalar field potential has a flat wing along which $V(\phi)$ is effectively a constant, which we denote by Λ . Equation (3.3) in such a two-component universe becomes

$$H^2 = \left(\frac{\kappa}{3} \rho + \frac{\Lambda}{3} \right) \left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa \rho_c} \right) - \frac{\mathbb{k}}{a^2}. \quad (3.5)$$

For a spatially closed universe, this equation can be recast as (2.3) with the effective potential

$$U(a) = -\frac{1}{2} a^2 \left(\frac{\kappa}{3} \rho + \frac{\Lambda}{3} \right) \left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa \rho_c} \right), \quad (3.6)$$

where

$$\frac{\kappa}{3} \rho = \frac{A}{a^l}, \quad l = 3(1+w), \quad w > -1/3. \quad (3.7)$$

The effective potential $U(a)$ in (3.6) exhibits a pair of extremes (a maximum and a minimum). The scale factor and the energy density corresponding to these extremes are given, respectively, by

$$a_{\pm}^l = \frac{3A(l-2)(\kappa\rho_c - 2\Lambda)}{4\Lambda(\kappa\rho_c - \Lambda)} \left[1 \pm \sqrt{1 - \frac{16(l-1)\Lambda(\kappa\rho_c - \Lambda)}{(l-2)^2(\kappa\rho_c - 2\Lambda)^2}} \right], \quad (3.8)$$

$$\rho_{\pm} = \frac{(l-2)}{4\kappa(l-1)} (\kappa\rho_c - 2\Lambda) \left[1 \pm \sqrt{1 - \frac{16(l-1)\Lambda(\kappa\rho_c - \Lambda)}{(l-2)^2(\kappa\rho_c - 2\Lambda)^2}} \right]. \quad (3.9)$$

Note that $a_- < a_+$ and $\rho_- < \rho_+$. The extremes can exist as long as the term under the square root in (3.8) and (3.9) remains positive, which imposes the following conditions on Λ :

$$\Lambda < \Lambda_{\text{crit}} = \frac{\kappa\rho_c}{2} \left[1 - \sqrt{\frac{q}{q+4}} \right], \quad \text{where } q = \frac{16(l-1)}{(l-2)^2}, \quad (3.10)$$

or

$$\Lambda > \kappa\rho_c. \quad (3.11)$$

For the first condition in (3.10), (a_-, ρ_+) is associated with the minimum of $U(a)$ while (a_+, ρ_-) is associated with the maximum of $U(a)$. On the other hand, for the second condition in (3.11) (i.e. $\Lambda > \kappa\rho_c$), only a_+ in (3.8) and ρ_- in (3.9) survive, and account for a minima in $U(a)$. However, according to (3.5), a spatially closed universe ($\mathbb{k} = 1$) does not support a solution with $\Lambda \geq \kappa\rho_c$. Hence this minima is shown, in Appendix A, to correspond to an ESU in a *spatially open* universe ($\mathbb{k} = -1$).

In this section our focus has been on the spatially closed universe ($\mathbb{k} = 1$) for which we can obtain static (or oscillating) solutions in the regime $\Lambda < \Lambda_{\text{crit}} < \kappa\rho_c$; i.e. according to (3.10). As noted in the previous section, a scalar field rolling along a flat potential with $V'(\phi) \equiv 0$, behaves just like stiff matter plus a cosmological constant. Our analysis in the following subsections will therefore focus on this important case.

3.1.1 The effective potential in the presence of stiff matter only

From (3.8) and (3.9), one can see that, as Λ tends to zero, a_+ approaches infinity and ρ_- declines to zero, but both a_- and ρ_+ remain finite. In other words, in the absence of the cosmological constant, the (unstable) maximum of the effective potential $U(a)$ disappears, while the stable minimum remains in place. This new feature of brane cosmology distinguishes it from GR, in which the effective potential has no extreme value when $\Lambda = 0$. As we shall see later, the persistence of a stable minimum of $U(a)$ in the absence of Λ carries over to other modified gravity models as well.

In the absence of the cosmological constant, and with a universe consisting only of stiff matter ($P = \rho$), the minimum of $U(a)$ is determined from (3.8) to be at

$$a_-^6 = \frac{15}{2} \frac{A}{\kappa \rho_c} . \quad (3.12)$$

As mentioned earlier in (2.5) – (2.7), the Einstein Static Universe (ESU) arises when $\dot{a} = 0$ and $\ddot{a} = 0$. As a result, the ESU is associated with an extremum of $U(a)$, so that $U'(a_E) = 0$ and $U(a_E) = -\mathbb{k}/2$ are jointly satisfied. In other words, a_E (the scale factor at the ESU) is the critical value of a_- for which the minimum of $U(a)$ lies on the $E = -\mathbb{k}/2$ line (see Fig. 7).

The stiff matter density at the minimum of $U(a)$ is determined from (3.9) to be

$$\rho_+ = \rho_E = \frac{2}{5} \rho_c , \quad (3.13)$$

where ρ_c is the braneworld constant defined in (3.3), (3.4). The fact that ρ_+ is independent of A implies that, for a universe which is oscillating about a_- , the density of matter at $U(a_-)$ *is the same* as in ESU. (This interesting result is *independent* of the equation of state of matter and holds for $l \geq 2$ in (3.7)). Hence we have denoted ρ_+ by ρ_E , the matter density at ESU. Note that for a closed universe ($\mathbb{k} = 1$) if $U(a_-) < -\frac{1}{2}$, then ESU will no longer be a solution of the field equations. Instead, the universe will oscillate around $U(a_-)$. With this result taken into account, the minimum in the effective potential $U(a)$ becomes

$$U(a_-) = -\frac{\kappa}{6} a_-^2 \rho_E \left(1 - \frac{\rho_E}{\rho_c} \right) , \quad (3.14)$$

while the ESU condition (2.6) for a closed universe ($\mathbb{k} = 1$) takes the form

$$U(a_E) = -\frac{1}{2} = -\frac{\kappa}{6} a_E^2 \rho_E \left(1 - \frac{\rho_E}{\rho_c} \right) . \quad (3.15)$$

Comparing (3.14) and (3.15) we easily find

$$U(a_-) = -\frac{1}{2} \frac{a_-^2}{a_E^2} . \quad (3.16)$$

By using (3.13) and (3.15), a_E is determined to be

$$a_E^2 = \frac{25}{2\kappa\rho_c} . \quad (3.17)$$

By using (2.3), with $\mathbb{k} = 1$, the condition to have a physical solution becomes

$$U(a_-) \leq -\frac{1}{2} \Rightarrow a_- \geq a_E \Rightarrow A \geq A_E , \quad (3.18)$$

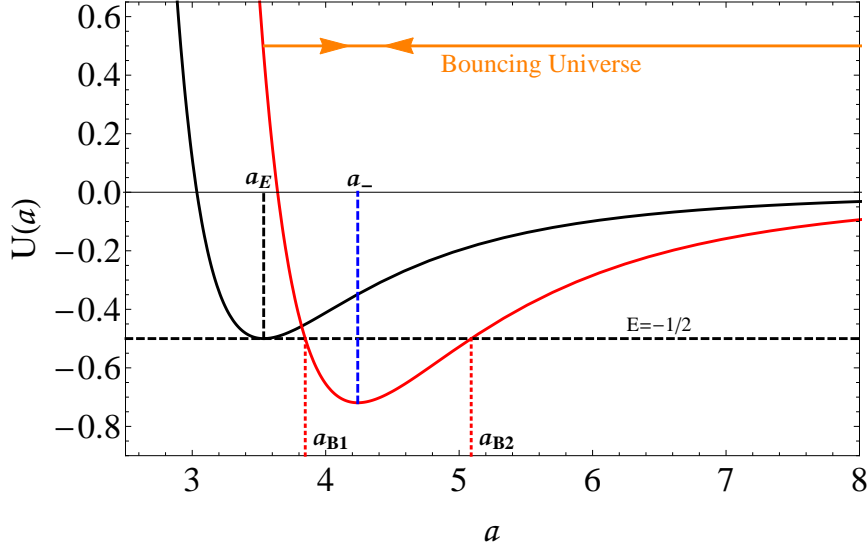


Figure 7. The effective potential is shown for a universe consisting of stiff matter with $P = \rho$. Two values of the parameter A are chosen. For $A = A_E$ (black curve), the minimum of $U(a)$ lies precisely on the $E = -1/2$ line and the solution is static at (a_E, ρ_E) which corresponds to the Einstein Static Universe (ESU). For $A > A_E$, the minimum of $U(a)$ shifts towards higher a and lies below the $E = -1/2$ line. In this case the ESU is absent and the universe oscillates around (a_-, ρ_-) , i.e. around the minimum of the effective potential (red curve). Note that the value of the energy density at the minimum of $U(a)$ is a fixed quantity and is given by (3.13). From the figure it is clear that the motion of a closed universe is always bounded, and the turning points of its scale factor are a_{B1} and a_{B2} . By contrast, a spatially flat/open universe bounces at small a but need not turn around and collapse. It is important to note that the above form of the potential $U(a)$ is robust and remains qualitatively unchanged if stiff matter is replaced by any other form of matter satisfying the SEC, i.e., for $\rho \propto A/a^l$ with $l \geq 2$ in (3.7). In our illustration we assume $\kappa = 1$, $\rho_c = 1$.

where A_E is the value of the parameter A in (3.7) for which the ESU conditions are met. In the last inequality of (3.18), we have used (3.12) and the fact that a_E is simply the value of a_- at ESU.

The effective potential is shown in Fig. 7 for a universe containing stiff matter. As $a \rightarrow 0$, the potential $U(a)$ diverges causing the universe to bounce and avoid the big bang singularity. For large a , the braneworld equations (3.3), (3.4) reduce to the GR limit ($\rho/\rho_c \ll 1$), and $U(a)$ asymptotically approaches zero. The motion of a spatially closed universe is therefore always bounded. The minima in $U(a)$ represent stable universes. Two values of the parameter A in (3.7) are considered. For $A = A_E$, the universe is static (ESU) at $a = a_E$ (black curve), while, for $A > A_E$, the universe oscillates around $a = a_-$ (red curve).

3.1.2 The effective potential in the presence of stiff matter and a cosmological constant

For stiff matter and Λ , the scale factor and stiff matter density at the extremes of $U(a)$ are determined by setting $l = 6$ in (3.8) and (3.9). Consequently, one obtains

$$a_{\pm}^6 = \frac{3A(\kappa\rho_c - 2\Lambda)}{\Lambda(\kappa\rho_c - \Lambda)} \left[1 \pm \sqrt{1 - \frac{5\Lambda(\kappa\rho_c - \Lambda)}{(\kappa\rho_c - 2\Lambda)^2}} \right] \quad (3.19)$$

and

$$\rho_{\pm} = \frac{1}{5\kappa} (\kappa\rho_c - 2\Lambda) \left[1 \pm \sqrt{1 - \frac{5\Lambda(\kappa\rho_c - \Lambda)}{(\kappa\rho_c - 2\Lambda)^2}} \right]. \quad (3.20)$$

As before, (a_-, ρ_+) is associated with the minimum of $U(a)$ while (a_+, ρ_-) is associated with the maximum of $U(a)$, and $a_- < a_+$, $\rho_- < \rho_+$. The unstable ESU associated with the maximum is GR-like and was previously encountered in section 2. Hereafter we shall focus on the stable ESU that is associated with the minimum of $U(a)$.

It follows from (3.19) that a_{\pm} increases as the parameter A increases, while ρ_{\pm} is constant for a given Λ . This empowers (3.16) and (3.18) to hold true even in the completely general case when $\Lambda \neq 0$; see (3.8) and (3.9). For a certain value of the parameter $A = A_E$, ESU conditions are satisfied and the corresponding scale factor can be determined using (2.6) and (3.20) to be

$$a_E^2 = \frac{3}{(\kappa\rho_+ + \Lambda) \left(1 - \frac{\rho_+}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} \right)}. \quad (3.21)$$

Note that the condition for the existence of two extreme values in $U(a)$, namely (3.10), is simplified to

$$\Lambda < \Lambda_{\text{crit}} = \kappa\rho_c \left[\frac{1}{2} - \frac{\sqrt{5}}{6} \right]. \quad (3.22)$$

The form of the effective potential is very sensitive to the value of Λ , as shown in Fig. 8(a). From this figure, we see that the potential $U(a)$ diverges as $a \rightarrow 0$, which allows the universe to bounce at small a and avoid the big bang singularity. For $\Lambda = 0$, there exists only a single minimum in $U(a)$, as discussed in section 3.1.1. The maximum, representing an unstable fixed point, appears when $\Lambda > 0$. As Λ increases, the maxima and the minima in $U(a)$ approach each other. They merge when $\Lambda = \Lambda_{\text{crit}}$, which results in an inflection point in the effective potential, see Fig. 8(b). For $\Lambda > \Lambda_{\text{crit}}$, the potential has no extreme value at finite a , which is indicative of the absence of fixed points in the corresponding dynamical system.¹

While figures 8(a) and 8(b) have been constructed for matter with $P = \rho$ (in view of possible links to the kinetic regime during pre-inflation), the main results of our analysis, based on (3.8), (3.9) and (3.10), should remain qualitatively true for any matter component with $w > -1/3$, with Λ_{crit} depending upon w .

3.1.3 Phase space analysis

The energy conservation equation (2.15) for stiff matter reads

$$\dot{\rho} = -6H\rho. \quad (3.23)$$

Differentiating (3.5) with respect to time leads to the Raychaudhuri equation:

$$\dot{H} = -\frac{\kappa}{3} \left[\left(2\rho - \frac{\Lambda}{\kappa} \right) - \frac{(\kappa\rho + \Lambda)}{\kappa\rho_c} \left(5\rho - \frac{\Lambda}{\kappa} \right) \right] - H^2. \quad (3.24)$$

¹ Note that $U(a)$ exhibits a minimum also for $\Lambda > \kappa\rho_c$. However in this case $U(a) > 0$ which does not allow an ESU in a spatially closed universe, but permits an ESU in an open universe, as shown in Appendix A.

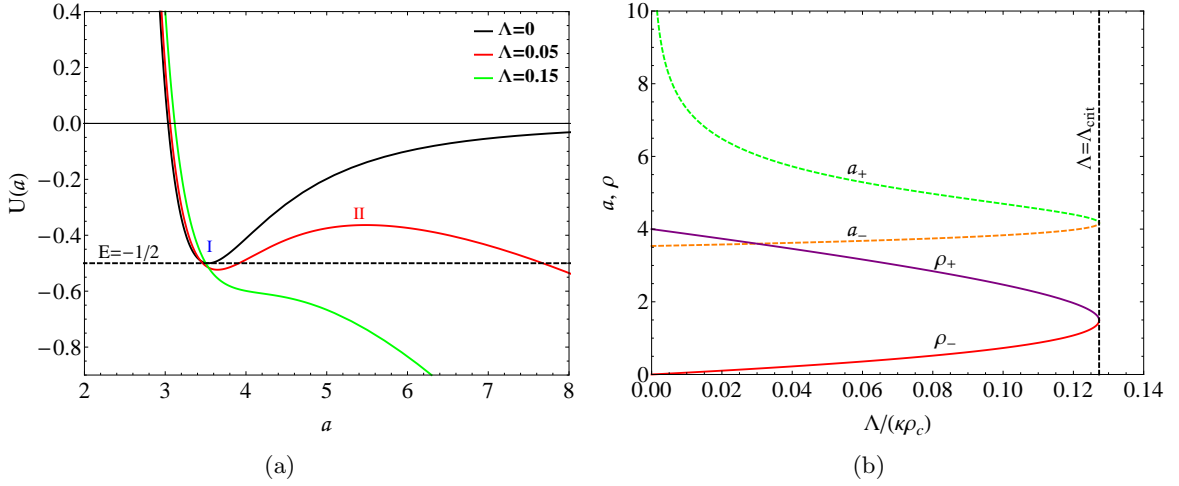


Figure 8. (a) The effective potential is plotted for different values of Λ for a two-component universe consisting of stiff matter and a cosmological constant. (i) For $\Lambda = 0$, there is only a single minimum representing a stable fixed point denoted by I (black); see section 3.1.1. (ii) For $\Lambda > 0$, there appears a maximum associated with the unstable fixed point, denoted by II (red). (iii) As Λ further increases, these two fixed points approach each other as illustrated in Fig. 8(b). At $\Lambda = \Lambda_{\text{crit}}$, they merge to produce an inflection point in $U(a)$, and, for $\Lambda > \Lambda_{\text{crit}}$, no fixed points are present in the system (green line in left panel). (b) The scale factor corresponding to the minimum (maximum) of $U(a)$, denoted by I (II) in Fig. 8(a) and by a_- (a_+) in (3.8), is plotted as a function of Λ in the right panel. The right panel also shows the value of the stiff matter density at the minimum (maximum) of $U(a)$, and denoted by ρ_+ (ρ_-) in (3.9). As Λ increases, the two fixed points, stable and unstable, move towards each other and merge at $\Lambda = \Lambda_{\text{crit}}$, beyond which no fixed point exists. Units of $\kappa = 1$ and $\rho_c = 1$ are assumed together with a suitable choice of A , for purposes of illustration. The unit along the y-axis is arbitrary.

The fixed points of the dynamical system described above are characterized by

$$\dot{\rho} = 0 \Rightarrow H = 0 \quad \text{and} \quad \dot{H} = 0 \Rightarrow \ddot{a} = 0, \quad (3.25)$$

precisely the same conditions as for ESU. For $\Lambda > 0$, the two fixed points in the (ρ, H) plane are $(\rho_+, 0)$ and $(\rho_-, 0)$, where ρ_{\pm} are given by (3.20). The condition for the existence of these fixed points was described earlier in (3.22). For $\Lambda = 0$, only one fixed point $(\rho_+, 0)$ exists. To analyse stability, the nonlinear dynamical system should be linearized using the *linearization theorem* (for instance, as described in [22]) near the two fixed points. The eigenvalues of the Jacobian matrix of the linearized system (as described in Appendix B) at the two fixed points are

$$\lambda_{I,II}^2 = 4\kappa\rho_{\pm} \left[1 - \frac{1}{\rho_c} \left(5\rho_{\pm} + \frac{2\Lambda}{\kappa} \right) \right], \quad (3.26)$$

where the indices I and II stand for the two fixed points $(\rho_+, 0)$ and $(\rho_-, 0)$, respectively. By using the values of ρ_{\pm} from (3.20), the properties of the eigenvalues at the two fixed points are listed in Table 1 for $\Lambda < \Lambda_{\text{crit}}$ in (3.22).

For the fixed point II $(\rho_-, 0)$, real eigenvalues with opposite sign imply that the fixed point is a saddle and therefore unstable. The eigenvalues for the fixed point I $(\rho_+, 0)$ are imaginary and complex conjugates of each other, suggesting that the fixed point in this case is of centre type (stable), as expected. But the linearization theorem does not guarantee this

| Fixed point | (ρ, H) | Eigenvalues | Stability |
|-------------|---------------|----------------------|----------------------------|
| I | $(\rho_+, 0)$ | $\lambda_I^2 < 0$ | Centre type stable point |
| II | $(\rho_-, 0)$ | $\lambda_{II}^2 > 0$ | Saddle type unstable point |

Table 1. Eigenvalues of the Jacobian matrix and stability of the fixed points for stiff matter and a cosmological constant.

for linearized systems of centre type [22], hence this result needs to be confirmed numerically. The phase portrait for general matter with $w > -1/3$ can be studied in a similar fashion. In general, the energy conservation equation becomes

$$\dot{\rho} = -3H\rho(1+w) , \quad (3.27)$$

while the Raychaudhuri equation is given by

$$\dot{H} = -\frac{\kappa}{3} \left[\frac{\rho}{2} (1+3w) - \frac{\Lambda}{\kappa} - \frac{(\kappa\rho + \Lambda)}{\kappa\rho_c} \left\{ (2+3w)\rho - \frac{\Lambda}{\kappa} \right\} \right] - H^2 . \quad (3.28)$$

Again, for $0 < \Lambda < \Lambda_{\text{crit}}$ the system possesses two fixed points I, II in the (ρ, H) plane, given by $(\rho_+, 0)$ and $(\rho_-, 0)$, respectively, where ρ_{\pm} and Λ_{crit} were determined in (3.9) and (3.10). The eigenvalues of the linearized system at the two fixed points are found to be

$$\lambda_{I,II}^2 = (1+3w)(1+w) \frac{\kappa}{2} \rho_{\pm} \left[1 - \frac{1}{\rho_c} \left\{ \frac{4(2+3w)}{(1+3w)} \rho_{\pm} + \frac{2\Lambda}{\kappa} \right\} \right] . \quad (3.29)$$

Using the values of ρ_{\pm} from (3.9), it can be shown that, for $(\rho_-, 0)$, we have $\lambda_{II}^2 > 0$, and real eigenvalues of opposite sign confirm the fixed point II to be a saddle. For I, the eigenvalues are again imaginary and complex conjugates of each other, given by $\lambda_I^2 < 0$. Although this suggests that the fixed point I is a centre, the linearization theorem does not assure this, hence this result needs to be confirmed numerically.

We have plotted the phase space for stiff matter and Λ using (3.23) and (3.24). Since we focus on the spatially closed case, we need to add the additional constraint

$$H^2 < \left(\frac{\kappa}{3} \rho + \frac{\Lambda}{3} \right) \left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} \right) , \quad (3.30)$$

which follows from (3.5). The resulting phase portrait is shown in Fig. 9(a) for $\Lambda = 0$. Note that only the centre type stable fixed point I exists in this case, as expected.

For $\Lambda > 0$, the unstable saddle point II appears along with the centre I. As Λ increases, these two fixed points move towards each other. The phase portrait for a typical value of Λ is illustrated in Fig. 9(b). When Λ reaches its critical value in (3.22), i.e., at $\Lambda = \Lambda_{\text{crit}}$, the stable and unstable points merge giving rise to an inflection point in the effective potential $U(a)$, as discussed earlier. For $\Lambda > \Lambda_{\text{crit}}$, fixed points are absent, and flows in the phase portrait are depicted in Fig. 9(c), suggesting that, after the bounce, the matter density declines monotonically as the universe expands.²

²Note that for $\Lambda > \kappa\rho_c$, the dynamical system admits a centre type stable fixed point given by $(\rho_-, 0)$ which has no physical significance for a spatially closed universe, in view of (3.30). However this fixed point is of great importance in constructing the emergent scenario in a spatially open universe as shown in Appendix A.

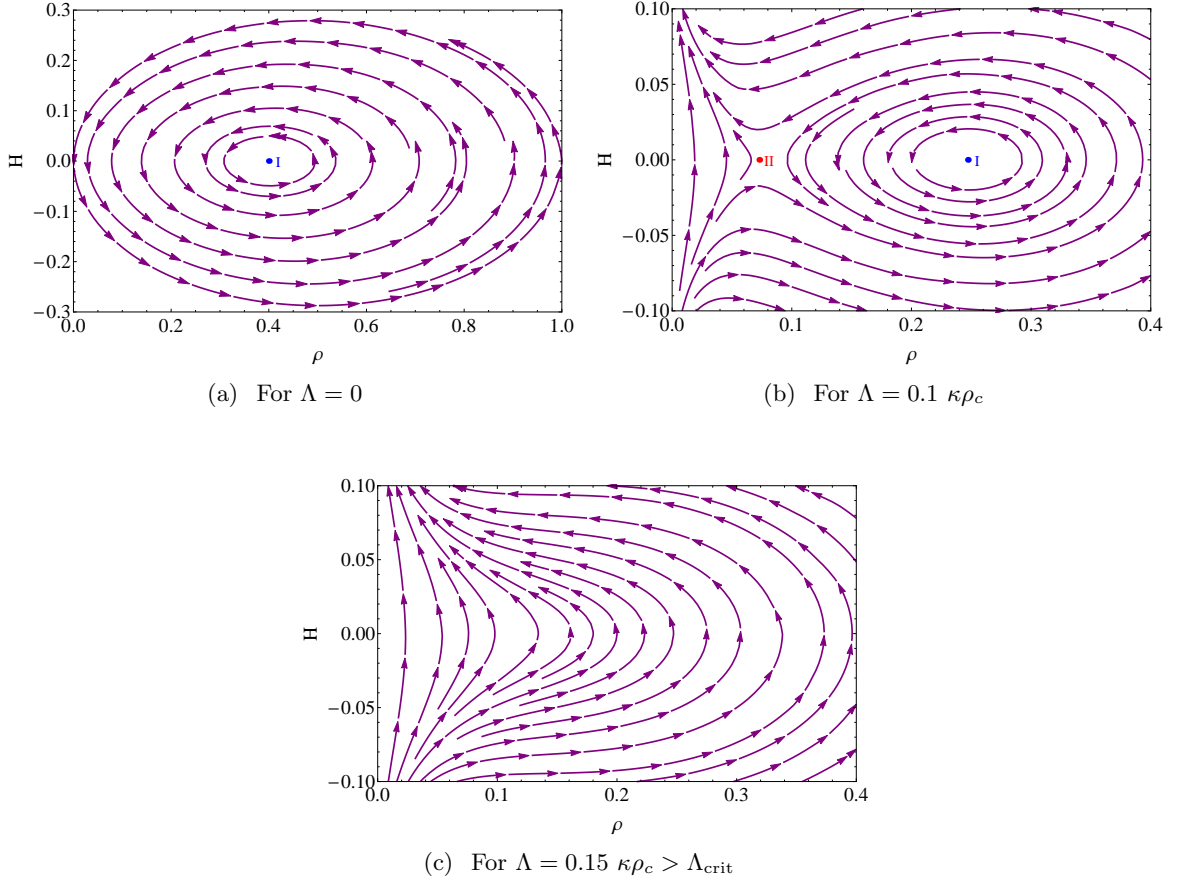


Figure 9. (a) Phase portrait of a universe consisting only of stiff matter. In this case, only the centre type fixed point I is present. The motion of the spatially closed braneworld is oscillatory and bounded. (b) Phase portrait for a closed braneworld consisting of stiff matter and a cosmological constant: $0 < \Lambda < \Lambda_{\text{crit}}$. For $\Lambda > 0$ the unstable saddle type fixed point II appears. As Λ increases the two fixed points, I and II, move towards each other and coincide at $\Lambda = \Lambda_{\text{crit}}$, giving rise to an inflection point in $U(a)$. (c) Phase portrait for stiff matter with $\Lambda > \Lambda_{\text{crit}}$. There is no fixed point in the dynamical system. In the numerical calculation we assume, for simplicity, that $\kappa = 1$ with $\rho_c = 1$.

3.1.4 Emergent scenario

The above discussion showed that: (i) the instability associated with ESU in the GR context can be avoided by studying such a model in the braneworld context; (ii) in this case, in addition to the unstable critical point II reminiscent of GR-based ESU, there appears a stable critical point I around which the universe can oscillate.

The construction of a realistic emergent cosmology requires that the universe be able to exit its oscillatory phase. In order to do this, the inflationary potential must be chosen judiciously, so that:

- [A] $V(\phi)$ have an asymptotically flat branch where $V(\phi) \simeq V_0$. This permits the appearance of the stable minimum I in the effective potential $U(a)$. At this minimum, which corresponds to the ESU, the field's kinetic energy and the scale factor of the universe

are given, respectively, by

$$\dot{\phi}^2 = \frac{2}{5} (\rho_c - 2V_0) \left[1 + \sqrt{1 - \frac{5V_0 (\rho_c - V_0)}{(\rho_c - 2V_0)^2}} \right] \quad (3.31)$$

and

$$a_E^2 = \frac{3M_p^2}{\rho(1 - \rho/\rho_c)} . \quad (3.32)$$

The corresponding value of ρ can be determined from (2.14a) using the value of $\dot{\phi}^2$ in (3.31). Note that (3.31) is identical to (3.20) for ρ_+ , while (3.32) is the same as (3.21).

[B] $V(\phi)$ should increase monotonically beyond some value of ϕ so that the effective cosmological constant, mimicked by V , increases with time. This allows the stable and unstable fixed points to merge and the ESU phase to end. Thereafter the universe inflates in the usual fashion. The potential described by (2.25), which is shown in Fig. 6 and was earlier discussed in [10] and [13], clearly satisfies this purpose.

It was earlier shown, in section 2, that the (flat) left wing of $V(\phi)$ can give rise to emergent cosmology in GR, provided inflation occurs from the (A) branch in Fig 6. For braneworld-based emergent cosmology one requires inflation to take place from the much steeper (B) branch of Fig. 6. This leads to a problem since along this branch the potential in (2.25) becomes an exponential, which is ruled out by recent CMB constraints [1]. Therefore, to examine this scenario further, we replace (2.25) by the following potential which adequately serves our purpose:

$$V(\phi) = V_0 \left[1 - \theta(\phi) \left(\frac{\phi}{M} \right)^\gamma \right]^2, \quad \gamma > 0, \quad (3.33)$$

where $\theta(\phi)$ is a step function: $\theta(\phi) = 0$ for $\phi < 0$, and $\theta(\phi) = 1$ for $\phi \geq 0$, which ensures that

$$V(\phi) = \begin{cases} V_0 & \text{for } \phi < 0, \\ 0 & \text{for } \phi = M, \\ V_0 \left(\frac{\phi}{M} \right)^{2\gamma} & \text{for } \phi \gg M. \end{cases} \quad (3.34)$$

Note that this potential qualitatively resembles the one shown in Fig. 6.

Once the scalar field begins to roll up its potential, $V(\phi)$ (mimicked by Λ in the previous section) increases, and the stable and unstable fixed points in Fig. 9(b) move towards each other (see Fig. 8(b)), merging when $V = V_{\text{crit}}$, where

$$V_{\text{crit}} = \rho_c \left[\frac{1}{2} - \frac{\sqrt{5}}{6} \right]. \quad (3.35)$$

For $V > V_{\text{crit}}$, there is no fixed point, and the universe escapes from attractor I.

For positive H in (2.13), the scalar field experiences a very large damping which causes it to stop climbing the potential $V(\phi)$. The inflationary regime commences once the scalar field begins to roll slowly down its potential.

The system of equations (2.13), (2.14) and (3.3), (3.4) has been integrated numerically in the context of spatially closed universe ($\mathbb{k} = 1$) using the potential given in (3.33). The

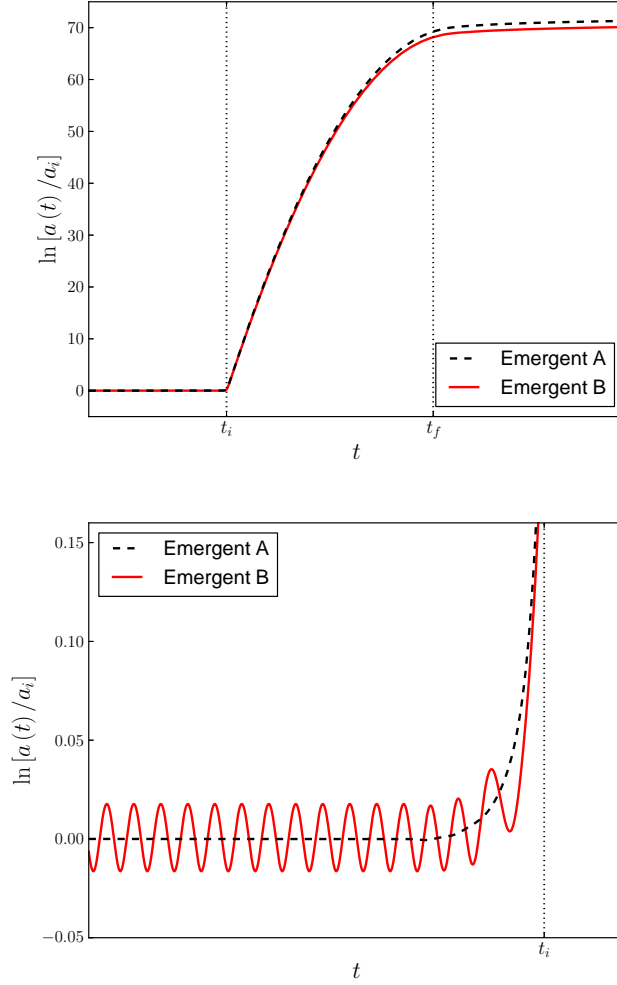


Figure 10. Top panel: Time evolution of the scale factor $a(t)$: t_i and t_f stand for the beginning and end of inflation, respectively. *Emergent A*: the universe was at ESU prior to inflation. *Emergent B*: the universe was perturbed slightly away from ESU before inflation. **Bottom panel** shows a magnified view of the top panel prior to inflation and demonstrates that universe B oscillates about the minimum of the effective potential. $\kappa = 1$ and $\rho_c = 1$ are assumed.

$\theta(\phi)$ function is realized numerically by $(1 + \tanh(\tau\phi))/2$ with a very large τ . Our results are shown in Fig. 10 for the following two cases: *Emergent A*: the universe was at ESU prior to the inflation; *Emergent B*: the universe was perturbed away from ESU before inflation. The beginning and end of inflation are denoted by t_i and t_f , respectively, on the time axis. The lower panel to Fig. 10 shows a zoomed-in view of the scale factor prior to inflation. This panel demonstrates that universe A stays at ESU (a_E) eternally in the past, whereas universe B exhibits oscillations around the minimum (a_-) of the effective potential $U(a)$. (Note that a_- is slightly displaced from a_E , as illustrated in Fig. 7 for $\Lambda = 0$.) The discerning reader may realize at this point that in Emergent A, the fine-tuning requirement on the field's kinetic energy, given by (3.31), is similar in spirit to that imposed in GR-based Emergent

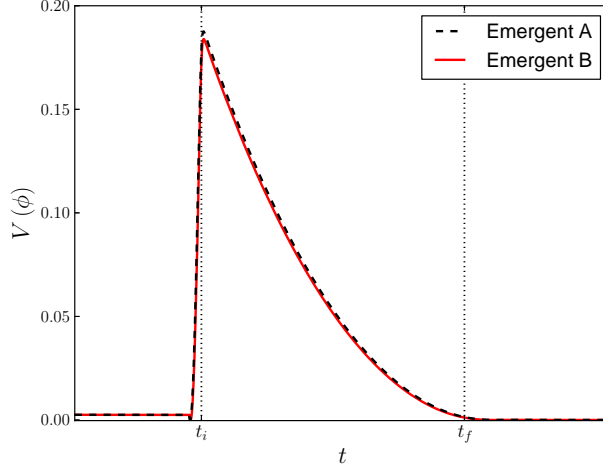


Figure 11. Time evolution of $V(\phi)$ in the emergent scenario described by (3.33). $\kappa = 1$, $\rho_c = 1$ are assumed.

cosmology, namely (2.17). In other words, given an effective potential with a minimum, it is much more ‘likely’ for the universe to oscillate about the ESU (Emergent B) than to sit exactly at the ESU fixed point (Emergent A). The time evolution of the inflaton potential $V(\phi)$ is illustrated in Fig. 11. The CMB constraints for this scenario can easily be satisfied, as shown in Appendix C.

3.2 Emergent scenario in Asymptotically Free Gravity

In this section we discuss the emergent scenario within the context of a theory of gravity which becomes *asymptotically free* at high energies. In this phenomenological model the gravitational constant depends upon the matter density as follows: $G(\rho) = G_0 \exp(-\rho/\rho_c)$. As a consequence, the FRW equations become

$$H^2 = \frac{\kappa}{3} \rho e^{-\rho/\rho_c} - \frac{\mathbb{k}}{a^2}, \quad (3.36)$$

$$\dot{H} = \frac{\kappa}{2} (1 + w) \rho e^{-\rho/\rho_c} \left[\frac{\rho}{\rho_c} - 1 \right] + \frac{\mathbb{k}}{a^2}, \quad (3.37)$$

where $\kappa = 8\pi G_0$ and G_0 is the asymptotic value of the gravitational constant: $G(\rho \rightarrow 0) = G_0$. It is easy to see that the low-energy limit of this theory is GR, while at intermediate energies $0 \ll \rho < \rho_c$ the field equations (3.36), (3.37) resemble those for the braneworld (3.3), (3.4). From (3.36), (3.37) we find that at large densities gravity becomes *asymptotically free*: $G(\rho) \rightarrow 0$ for $\rho \gg \rho_c$. In this case:

- The universe will bounce if $\mathbb{k} = 1$.
- The universe will ‘emerge’ from Minkowski space ($\dot{a} = 0, \ddot{a} = 0$ as $a \rightarrow 0$) if $\mathbb{k} = 0$.
- The universe will ‘emerge’ from the Milne metric ($a \propto t$ as $a \rightarrow 0$) if $\mathbb{k} = -1$.

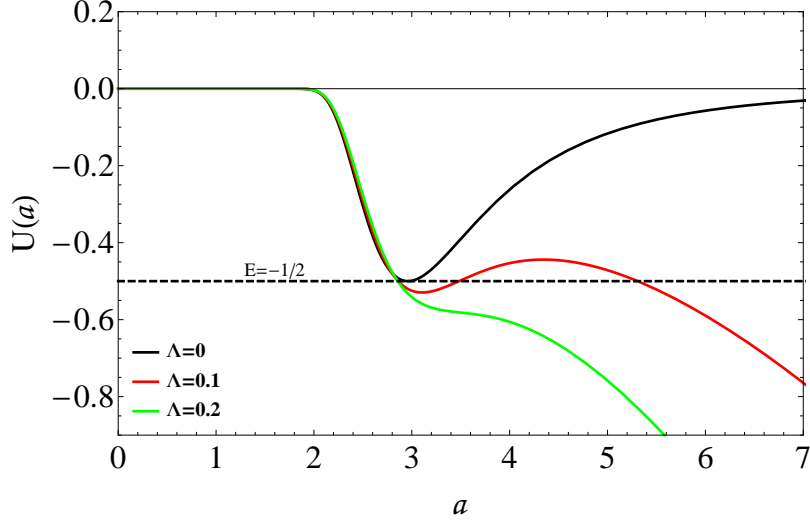


Figure 12. The effective potential, $U(a)$, is shown for *Asymptotically free gravity* for various values of Λ (in units $\kappa = 1$ and $\rho_c = 1$). For $\Lambda = 0$ there is only a single minima in $U(a)$, shown by the black line. A maxima appears when $\Lambda > 0$ (red line). As Λ increases the two extrema move towards each other and merge when $\Lambda = \Lambda_{\text{crit}}$ which results in an inflection point in $U(a)$. For $\Lambda > \Lambda_{\text{crit}}$ there is no extrema in $U(a)$ (green line).

In all three cases the big bang singularity is absent.
The effective potential for this theory is

$$U(a) = -\frac{\kappa}{6}a^2\rho e^{-\rho/\rho_c}, \quad (3.38)$$

where the evolution of ρ is given in (3.7). In order to link the emergent scenario with inflation we shall consider ρ to represent the density of the inflaton field. Recall that the inflaton moving along a flat direction ($V' = 0$) behaves like a fluid consisting of two non-interacting components, namely stiff matter and the cosmological constant. Hence ρ in (3.36), (3.37) & (3.38) is effectively replaced by $\rho_\phi \equiv \rho_{\text{stiff}} + \rho_\Lambda$ where $\rho_{\text{stiff}} \propto a^{-6}$ and $\rho_\Lambda = \Lambda/\kappa$.

If $\Lambda = 0$ then the effective potential in (3.38) exhibits a single minimum shown by the black line in Fig. 12. As noted in section 2, the existence of an ESU implies the simultaneous implementation of the following conditions:

1. $\ddot{a} = 0 \Rightarrow U'(a_E) = 0$,
2. $\dot{a} = 0 \Rightarrow U(a_E) = -\mathbb{k}/2$.

Focussing on a closed universe ($\mathbb{k} = 1$) dominated by stiff matter ($\Lambda = 0$) one finds that the scale factor and density associated with ESU are given by

$$\rho_E = \frac{2}{3}\rho_c, \quad a_E^2 = \frac{9}{2\kappa\rho_c}e^{2/3}. \quad (3.39)$$

Including $\Lambda > 0$, one finds that the form of $U(a)$ changes to accommodate *both* a maxima and a minima. The scale factor and density associated with these extrema are given by

$$a_{\pm}^6 = \frac{3A}{2\Lambda\kappa\rho_c} (2\kappa\rho_c - 3\Lambda) \left[1 \pm \sqrt{1 - \frac{12\Lambda\kappa\rho_c}{(2\kappa\rho_c - 3\Lambda)^2}} \right], \quad (3.40)$$

$$\rho_{\pm}^6 = \frac{(2\kappa\rho_c - 3\Lambda)}{6\kappa} \left[1 \pm \sqrt{1 - \frac{12\Lambda\kappa\rho_c}{(2\kappa\rho_c - 3\Lambda)^2}} \right]. \quad (3.41)$$

Again (a_-, ρ_+) corresponds to the minimum, while (a_+, ρ_-) is associated with the maximum in $U(a)$. Note that ρ_{\pm} are independent of the value of the parameter A defined in (3.7). This implies that for a universe which oscillates about a_- , the matter density at $U(a_-)$ is the same as in the ESU. Recall that this situation was earlier encountered for the Braneworld in section 3.1 and implies that equations (3.16) and (3.18) hold in the present case too. Once more we shall focus on the stable ESU corresponding to the minima. The ESU is given by the scale factor

$$a_E^2 = \frac{3}{(\kappa\rho_+ + \Lambda)} \exp \left[\frac{\kappa\rho_+ + \Lambda}{\kappa\rho_c} \right]. \quad (3.42)$$

As Λ increases the minima and maxima in (3.40) approach each other, merging when

$$\Lambda = \Lambda_{\text{crit}} = \frac{4}{3}\kappa\rho_c \left(1 - \frac{\sqrt{3}}{2} \right), \quad (3.43)$$

which results in an inflection point in $U(a)$. The effective potential in *Asymptotically free gravity* shown in Fig. 12 resembles that in the braneworld model³ in Fig. 8. We therefore find that, as in the braneworld, an emergent scenario can be supported by the inflaton potential in (3.33).

3.3 Emergent scenario in Loop Quantum Cosmology

In Loop Quantum Cosmology (LQC), the FRW equation for a spatially closed universe ($\mathbb{k} = 1$) consisting of matter which satisfies the SEC and Λ is [12, 14]

$$H^2 = \left(\frac{\kappa}{3}\rho + \frac{\Lambda}{3} - \frac{1}{a^2} \right) \left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} + \frac{3}{\kappa\rho_c a^2} \right), \quad (3.44)$$

where $\rho_c \sim M_p^4$. In this case the effective potential depends on the curvature, and for $\mathbb{k} = 1$ one determines it from (2.3) to be

$$U(a) = - \left(\frac{\kappa}{6}\rho a^2 + \frac{\Lambda}{6}a^2 - \frac{1}{2} \right) \left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} + \frac{3}{\kappa\rho_c a^2} \right) - \frac{1}{2}, \quad (3.45)$$

where ρ is given by (3.7). The ESU conditions (2.5) provide two possibilities for an *Einstein Static Universe* in this scenario, which have been listed in Table 2. An unstable ESU appears for $\Lambda > 0$ which is identical to the ESU in GR, earlier discussed in section 2. The stable ESU appears due to LQC modifications and only exists for $\Lambda > \kappa\rho_c$ [14].

As discussed earlier, a scalar field driven scenario, in which the scalar rolls along a flat potential ($V' = 0$), is equivalent to stiff matter together with a cosmological constant Λ . The effective potential in this case is shown for various values Λ in Fig. 13. It is worth noting that every extremum (maxima or minima) in $U(a)$ does not necessarily result in an ESU since every solution of $\ddot{a} = 0$ may not support $\dot{a} = 0$.

³Note that unlike the braneworld case in section 3.1, this asymptotically free gravity model admits no minima in $U(a)$ for large Λ , i.e. for $\Lambda > \kappa\rho_c$.

| Fixed point (ESU) | Λ | ρ | a_E |
|-------------------|--------------------------|---|--|
| Unstable | $\Lambda > 0$ | $\rho_{GR} = \frac{2\Lambda}{\kappa(1+3w)}$ | $a_{GR}^2 = \frac{1+3w}{\Lambda(1+w)}$ |
| Stable | $\Lambda > \kappa\rho_c$ | $\rho_{LQC} = \frac{2(\Lambda - \kappa\rho_c)}{\kappa(1+3w)}$ | $a_{LQC}^2 = \frac{1+3w}{(\Lambda - \kappa\rho_c)(1+w)}$ |

Table 2. Density and scale factor for the *Einstein Static Universe* (ESU) in LQC. The stable ESU is denoted by ‘LQC’ while the unstable fixed point resembles the ‘GR’ case.

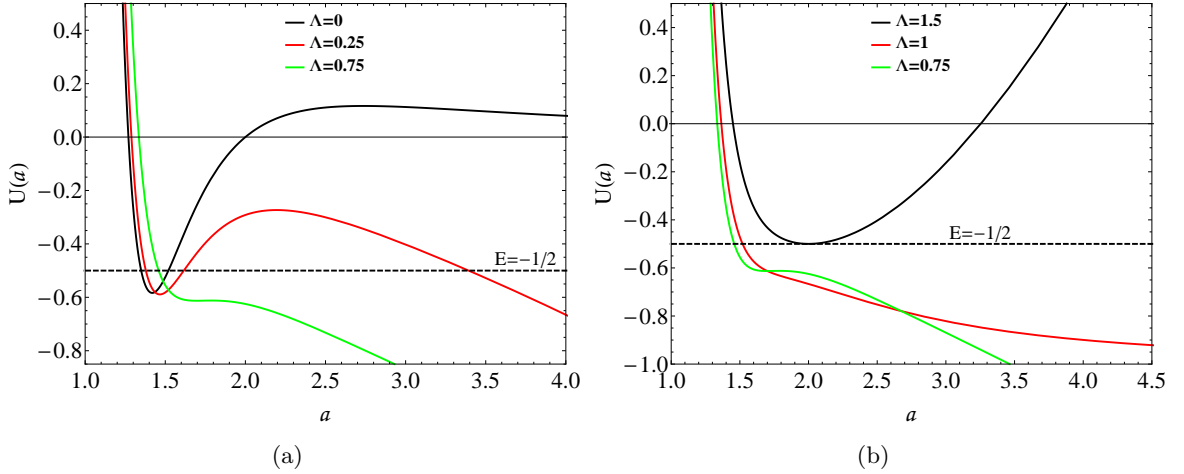


Figure 13. The effective potential for LQC is plotted for *small* values of Λ in the left panel (a), and for *large* values of Λ in the right panel (b). In both cases one assumes that the universe consists of stiff matter in addition to the cosmological constant. This combination mimics the behavior of a scalar field rolling along a *flat* direction in the potential ($V' = 0$) as described by (3.33) or (2.16). We choose a typical value of the parameter A in (3.7) while assuming $M_p = 1$ with $\rho_c \sim M_p^4$. The left panel indicates that inflation could proceed via the potential (3.33) illustrated in Fig. 6. Whereas the right panel supports inflation described by (2.16) and illustrated in Fig. 2.

Figures 13(a) and 13(b) demonstrate that the emergent scenario in LQC can arise in two distinct ways:

1. As shown in Fig. 13(a) a minimum in $U(a)$ can exist for small values of Λ . As Λ increases this minimum gets destabilized. This indicates that an inflaton potential such as (3.33) discussed earlier in the braneworld context, could also give rise to an emergent scenario in LQC. Note that unlike the braneworld case, a stable ESU does not exist in LQC for small Λ – see Table 2. However this does not prevent the universe from *oscillating about* the minimum of $U(a)$, thereby giving rise to a stable emergent scenario. It is easy to see that since $\rho \ll \rho_c$ during inflation, the CMB constraints on the parameters of the inflaton potential in (3.33) will be similar to those discussed in Appendix C in the braneworld context.
2. The presence of a minimum in $U(a)$ is illustrated in Fig. 13(b) for *large* values of Λ : $\Lambda > \kappa\rho_c$. This minimum is associated with a stable ESU as demonstrated in Table 2.

As Λ *decreases* this minimum gets destabilized. This suggests that a potential such as (2.16), earlier discussed in the GR context, could give rise to an emergent scenario in LQC.

Possibility 2 might however be problematic in two respects:

As noted in Table 2, the value of $V_0 (\equiv \Lambda/\kappa)$ in the flat wing of the emergent cosmology potential must be larger than $\rho_c \sim M_p^4$ in order for LQC effects to successfully drive emergent cosmology.

- (i) $V_0 > M_p^4$ might question the semi-classical treatment pursued by us in this section.
- (ii) While the potential (2.16) can successfully drive an emergent scenario in LQC, CMB bounds derived in section 2.2 suggest $V_0 < 10^{-8} M_p^4$, which conflicts with the LQC requirement ⁴ $V_0 \gtrsim M_p^4$.

While (i) lies outside the scope of the present paper, we demonstrate in Appendix D that CMB constraints can be satisfied even with $V_0 > M_p^4$ provided the scalar field Lagrangian possesses *non-canonical* kinetic terms [23].

All of the emergent scenarios discussed in this section passed through a prolonged (formally infinite) duration quasi-static stage during which the universe was located either at the Einstein Static fixed point (ESU) or oscillated around it. In the next section we examine the semi-classical properties of the oscillatory universe, focusing especially on its impact on graviton production.

4 Graviton production in an oscillatory universe

In a flat FRW universe, each of the two polarization states of the graviton behaves as a massless minimally coupled scalar field [24]. This is also true for the massless graviton modes in the higher-dimensional theories (see, e.g., [25] for the case of braneworld model). While the conformal flatness of the FRW space-time ensures that the creation of conformally coupled fields (including photons) does not happen, no such suppression mechanism exists for fields that couple non-conformally to gravity [26, 27]. Indeed, it is well known that gravitons are generically created in a FRW universe, and this effect has been very well studied in the context of inflation [28–30]. One, therefore, naturally expects gravity waves to be created in an oscillating universe such as the one examined in the previous section, in the context of emergent cosmology. That this is indeed the case will be demonstrated below.

In the emergent scenario, the universe oscillates around a fixed value of the scale factor for an indefinite amount of time before (gradually changing value of) the potential ends the oscillatory regime and leads to inflation. While oscillating, the universe produces gravitons by a quantum-mechanical process. A large graviton density (compared with that of the existing matter), can disrupt the oscillatory regime making it difficult for emergent cosmology to be past-eternal. Here we investigate this issue in the context of the braneworld scenario described in section 3.1.

⁴While the CMB bounds in section 2.2 were derived using GR, this would also be a good approximation to the LQC case in which $\rho \ll \rho_c$ once inflation commences, and LQC effects can be ignored.

4.1 Resonant particle production

Tensor metric perturbations (or gravity waves) in general relativity are described by the quantities h_{ij} defined as

$$\delta g_{ij} = a^2 h_{ij}, \quad i, j = 1, 2, 3. \quad (4.1)$$

The tensor h_{ij} is transverse and traceless and obeys the following equations [31]:

$$h''_{ij} + 2\mathcal{H}h'_{ij} - (\nabla^2 - 2\mathbb{k})h_{ij} = 0, \quad (4.2)$$

where $\mathcal{H} = a'/a$, and $\mathbb{k} (= 0, \pm 1)$ is the spatial curvature of the FRW universe. Here, the prime denotes differentiation with respect to the conformal time η . For a spatially closed universe ($\mathbb{k} = 1$), we make the rescaling $h_{ij} = \chi_{ij}/a$ and pass to the generalized Fourier transform on the three-sphere

$$\chi_{ij}(\eta, x) = \sum_{n\ell} \chi_{n\ell}(\eta) Y_{ij}^{n\ell}(x), \quad (4.3)$$

where $Y_{ij}^{n\ell}(x)$ are the normalized transverse and traceless tensor eigenfunctions of the Laplacian operator on a unit three-sphere. They are labeled by the main quantum number n and by the collective quantum numbers $\ell = \{p, l, m\}$, which have the following meaning [32]:

$$\begin{aligned} n &= 3, 4, 5, \dots && \text{(main quantum number)}, \\ p &= 1, 2 && \text{(polarization)}, \\ l &= 2, 3, \dots, n-1 && \text{(angular momentum)}, \\ m &= -l, -l+1, \dots, l && \text{(angular momentum projection)}. \end{aligned} \quad (4.4)$$

The eigenvalues $-k_n^2$ of the Laplacian operator ∇^2 for transverse traceless tensor modes on a unit three-sphere depend only on n and are given by [32]

$$k_n^2 = n^2 - 3, \quad n = 3, 4, 5, \dots \quad (4.5)$$

Equation (4.2) then leads to the following equations for the Fourier coefficients $\chi_{n\ell}(\eta)$:

$$\chi''_{n\ell} + \left(k_n^2 + 2 - \frac{a''}{a} \right) \chi_{n\ell} = 0. \quad (4.6)$$

Sufficiently close to the minimum of the effective potential, the universe can exhibit oscillatory motion. Subject to a small perturbation, the oscillatory motion with frequency ω satisfies

$$a(t) = a_- + \delta a \cos \omega t, \quad (4.7)$$

where the frequency is given by

$$\omega^2 = \frac{d^2 U(a_-)}{da^2}, \quad (4.8)$$

and the amplitude is

$$\delta a^2 = -\frac{2}{\omega^2} \left[U(a_-) + \frac{1}{2} \right]. \quad (4.9)$$

To be able to use (4.2) and (4.6), we express this motion in terms of the conformal time η :

$$\eta = \int \frac{dt}{a(t)} = \int \frac{dt}{a_- + \delta a \cos \omega t} \approx \frac{t}{a_-} \quad \text{for} \quad \frac{\delta a}{a_-} \ll 1. \quad (4.10)$$

Now, the functions a and a''/a can be calculated as

$$a(\eta) = a_- + \delta a \cos \zeta \eta , \quad (4.11a)$$

$$\frac{a''}{a} = -\frac{\delta a}{a_-} \zeta^2 \cos \zeta \eta , \quad (4.11b)$$

where

$$\zeta = a_- \omega \quad (4.12)$$

is the frequency in the conformal time η .

We would like to apply the theory of parametric resonance, as described in [33], to equation (4.6). This equation has the general form that was under consideration in [33]:

$$\chi_{n\ell}'' + [\zeta_n^2 + \epsilon g(\zeta \eta)] \chi_{n\ell} = 0 , \quad (4.13)$$

where $g(x)$ is a 2π -periodic function, and ϵ is a convenient small parameter. In our case, we have

$$\zeta_n^2 = k_n^2 + 2 = n^2 - 1 , \quad n = 3, 4, 5, \dots , \quad (4.14)$$

$$g(x) = \zeta^2 \cos x , \quad (4.15)$$

and

$$\epsilon = \frac{\delta a}{a_-} . \quad (4.16)$$

Equation (4.13) with the function $g(x)$ given by (4.15) is just the Mathieu equation. It is known that the first resonance band for this equation, which is dominant for small values of ϵ , lies in the neighbourhood of the frequency

$$\zeta_{\text{res}} = \frac{\zeta}{2} , \quad (4.17)$$

and the resonant amplification takes place for eigenfrequencies satisfying the condition

$$\Delta_n^2 < g_1^2 . \quad (4.18)$$

Here,

$$\Delta_n = \frac{1}{\epsilon} (\zeta_n^2 - \zeta_{\text{res}}^2) , \quad (4.19)$$

and

$$g_1 = \frac{1}{2} \zeta^2 \quad (4.20)$$

is the Fourier amplitude of the harmonic function $g(x)$.

Within the resonance band (4.18), particle production proceeds exponentially with time, so that the number of quanta in the mode grows as

$$N_n = \frac{1}{1 - \Delta_n^2/g_1^2} \sinh^2 \mu_n \eta , \quad (4.21)$$

where

$$\mu_n = \frac{\epsilon}{\zeta} \sqrt{g_1^2 - \Delta_n^2} . \quad (4.22)$$

The condition (4.18) for instability in the first resonance band can be expressed as

$$\left| \zeta_n^2 - \left(\frac{\zeta}{2} \right)^2 \right| < \frac{\epsilon \zeta^2}{2} . \quad (4.23)$$

4.1.1 Radiation-dominated universe with $\Lambda = 0$

From the viewpoint of the effective potential $U(a)$, the created gravitons behave as radiation. Therefore, for simplicity of the analysis, we consider a radiation-dominated universe, in which the produced gravitons just increase the existing radiation energy density (i.e., their effect will consist in the increase of A in (3.7)). For further simplification, we first consider the case with no cosmological constant. The frequency of oscillations (with respect to the cosmic time t) turns out to be constant in this case :

$$\omega^2 = \frac{d^2 U(a_-)}{da^2} = \frac{4}{9} \kappa \rho_c = \frac{4}{9M_p^2} \rho_c . \quad (4.24)$$

Therefore, adding more to the existing radiation energy density (increasing the quantity A in (3.7)) does not affect ω . The energy densities at the extremes of the effective potential are also constant (independent of A in (3.7)) :

$$\rho_+ = \frac{1}{3} \rho_c , \quad \rho_- = 0 . \quad (4.25)$$

The second equation implies that there are no maxima in the effective potential (or the maximum is reached as $a \rightarrow \infty$).

From (2.6) and (4.25), the scale factor and the quantity ζ can be evaluated for the ESU :

$$a_E^2 = \frac{27}{2\kappa\rho_c} = \frac{27}{2} \frac{M_p^2}{\rho_c} , \quad (4.26)$$

$$\zeta_E^2 = \omega^2 a_E^2 = 6 . \quad (4.27)$$

Eventually, the terms appearing in (4.13) can be calculated as

$$\zeta^2 = \omega^2 a_-^2 = 6 \left(\frac{a_-^2}{a_E^2} \right) \geq 6 \quad (4.28)$$

and

$$\epsilon^2 = \left(\frac{\delta a}{a_-} \right)^2 = \frac{1}{6} - \frac{9M_p^2}{4\rho_c a_-^2} = \frac{1}{6} \left(1 - \frac{a_E^2}{a_-^2} \right) \leq \frac{1}{6} . \quad (4.29)$$

Although ω is independent of graviton production, we can see from (3.8) that a_- increases as more and more gravitons are produced. This leads to an increase both in ζ and in ϵ (although ϵ has an asymptotic saturation value $1/\sqrt{6}$). Now, we assume

$$x = \left(\frac{a_-}{a_E} \right)^2 \geq 1 . \quad (4.30)$$

In this case, the resonance condition (4.23) has the form

$$\left(\zeta_n^2 - \frac{3}{2}x \right)^2 < \frac{3}{2}x(x-1) . \quad (4.31)$$

In Table 3, the resonance intervals of the quantity a_-/a_E along with the corresponding intervals of ϵ are shown for several lower modes (ζ_n^2 is given by (4.14) and (4.5)) by solving the inequality (4.31) in terms of x . As a graviton mode gets excited, the quantity a_-/a_E

| ζ_n^2 | a_-/a_E | ϵ |
|-------------|-------------|-------------|
| 8 | 1.784–5.179 | 0.338–0.401 |
| 15 | 2.396–7.229 | 0.371–0.404 |
| 24 | 3.007–9.217 | 0.385–0.406 |

Table 3. Resonance intervals of the quantities a_-/a_E and ϵ for several lower modes in a radiation-dominated universe.

increases due to the growth of A in (3.8). The resonant intervals of a_-/a_E for neighboring values of n overlap, as demonstrated in Table 3 for several lowest values of n . It is clear then that, if the initial value of a_-/a_E is sufficiently large, so that it falls in any of the resonance regions, then this will lead to excitation of all modes, one by one, resulting in a monotonous increase in a_-/a_E due to graviton production. In this case, the oscillatory regime is unstable with respect to graviton production. However, if the initial amplitude of oscillations is sufficiently small, so that $\epsilon < 0.338$, or $a_-/a_E < 1.784$, then no graviton mode is in the resonance initially, and past eternal oscillation of the universe is stable with respect to resonant production of gravitons.

Although the values of ϵ in Table 3 turn out to be not much smaller than unity, they are still considerably small, and we believe that our analysis in this case is qualitatively correct.

4.1.2 Stiff-matter dominated universe

For a constant scalar field potential $V(\phi)$, the kinetic energy density $\frac{1}{2}\dot{\phi}^2$ in (2.14) evolves as a^{-6} , similarly to the energy density of stiff matter. Therefore, when dealing with inflation based on a scalar field, it is a good idea to consider the case of a stiff-matter dominated universe. It was observed previously that the value a_- of the scale factor corresponding to the minimum of the effective potential actually increases as more and more gravitons are added to the existing stiff-matter energy density. This was also verified numerically. Below, we analyze both the simple case with $\Lambda = 0$ and the more realistic case with non-zero Λ .

[A] No cosmological constant, $\Lambda = 0$

The analysis for stiff-matter dominated universe is carried out in a manner similar to that of the universe filled with radiation that was under investigation in section 4.1.1. Here, the frequency in cosmic time ω again turns out to be constant (independent of A in (3.7)) and is given by

$$\omega^2 = \frac{d^2 U(a_-)}{da^2} = \frac{8}{5} \kappa \rho_c = \frac{8}{5 M_p^2} \rho_c. \quad (4.32)$$

The value ρ_E of the stiff matter energy density at the minimum of the effective potential was already calculated in (3.13). Using the value of a_E from (3.17), we figure out the value of ζ^2 at the ESU:

$$\zeta_E^2 = \omega^2 a_E^2 = 20. \quad (4.33)$$

Again, the terms appearing in (4.23) can be calculated as

$$\zeta^2 = \omega^2 a_-^2 = 20 \left(\frac{a_-^2}{a_E^2} \right) \geq 20 \quad (4.34)$$

and

$$\epsilon^2 = \frac{1}{\omega^2} \left(\frac{2}{25} \kappa \rho_c - \frac{1}{a_-^2} \right) = \frac{1}{20} \left(1 - \frac{a_E^2}{a_-^2} \right) \leq \frac{1}{20} . \quad (4.35)$$

Introducing the quantity x as in (4.30), and using (4.34) and (4.35), we present the resonance condition (4.23) in the form

$$(\zeta_n^2 - 5x)^2 < 5x(x - 1) . \quad (4.36)$$

| ζ_n^2 | a_-/a_E | ϵ |
|-------------|-------------|-------------|
| 8 | 1.146–1.561 | 0.109–0.172 |
| 15 | 1.5–2.236 | 0.167–0.200 |
| 24 | 1.867–2.875 | 0.189–0.210 |

Table 4. Resonance intervals of the quantities a_-/a_E and ϵ for several lower modes in a stiff-matter dominated universe.

The resonance intervals of a_-/a_E and ϵ , given by inequality (4.36), are listed for several lowest modes in Table 4. Again, the overlapping of the resonance bands indicates that, once a graviton mode is excited, the monotonous growth in a_-/a_E will lead to resonant excitation of the next modes. However, for $a_-/a_E < 1.146$, or $\epsilon < 0.109$, the past eternal oscillation of the universe is stable with respect to resonant production of gravitons.

[B] Stiff matter with cosmological constant Λ

If the value of the flat wing of the scalar-field potential is non-negligible, then, in our model of stiff matter, we must also introduce the cosmological constant Λ . As before, the frequency ω is independent of the quantity A in (3.7),

$$\omega^2 = \frac{d^2 U(a_-)}{da^2} = -\frac{10}{3} \kappa \rho_+ \left(1 - \frac{2\Lambda}{\kappa \rho_c} \right) + \frac{55}{3} \frac{\kappa \rho_+^2}{\rho_c} - \frac{\Lambda}{3} \left(1 - \frac{\Lambda}{\kappa \rho_c} \right) , \quad (4.37)$$

while ρ_+ and a_E are already given in (3.20) and (3.21), respectively. The terms appearing in (4.23) can be expressed more generally as

$$\zeta^2 = \omega^2 a_-^2 = \zeta_E^2 \left(\frac{a_-^2}{a_E^2} \right) \geq \zeta_E^2 \quad (4.38)$$

and

$$\epsilon^2 = \frac{1}{\omega^2} \left(\frac{1}{a_E^2} - \frac{1}{a_-^2} \right) = \frac{1}{\zeta_E^2} \left(1 - \frac{a_E^2}{a_-^2} \right) \leq \frac{1}{\zeta_E^2} . \quad (4.39)$$

In deriving (4.39), we used equation (3.16).

For a general value of Λ , the value of ζ_E^2 cannot be evaluated analytically. For $\Lambda = 0$, the calculation is presented earlier in this subsection 4.1.2. In the opposite limit $\Lambda \rightarrow \kappa \rho_c (1/2 - \sqrt{5}/6)$, we have

$$a_E^2 = \frac{15}{\kappa \rho_c} , \quad \omega^2 = 0 , \quad \zeta_E^2 = 0 . \quad (4.40)$$

It is impossible to satisfy the resonance condition (4.23) in this limit of Λ , hence no resonant graviton production takes place. Now, as Λ increases a_- also increases while ω decreases.

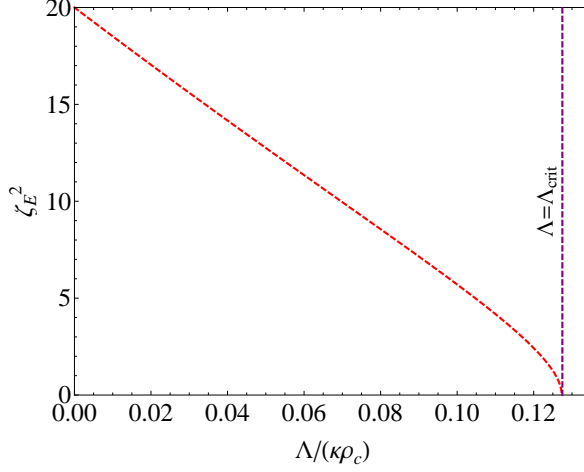


Figure 14. ζ_E^2 vs Λ in the natural units $\kappa = 1$ and $\rho_c = 1$.

Fig. 14 shows that ζ_E^2 actually decreases with increasing Λ , and ζ_E^2 goes to zero as Λ approaches the limiting value specified in (3.22).

For convenience, along with the variable x defined in (4.30), we also introduce the variable $y = \zeta_E^2/4$. Then

$$\frac{\zeta^2}{4} = \frac{\zeta_E^2}{4} \left(\frac{a_-^2}{a_E^2} \right) = xy, \quad (4.41)$$

and, for a general ζ_E , the resonance condition (4.23) becomes

$$(\zeta_n^2 - yx)^2 < yx(x - 1). \quad (4.42)$$

Solving the inequality with respect to x for a specified value ζ_n^2 , we get the allowed range of x to excite the respective mode. The point $y = 1$ is critical, so one needs to distinguish between two cases.

(i) For $y > 1$, the range of allowed x lies in the interval (x_-, x_+) , where the end points are given by

$$x_{\pm} = \left(\frac{a_-}{a_E} \right)^2 = \frac{2\zeta_n^2 - 1}{2(y - 1)} \left[1 \pm \sqrt{1 - \frac{4\zeta_n^4(y - 1)}{(2\zeta_n^2 - 1)^2 y}} \right]. \quad (4.43)$$

(ii) For $0 < y < 1$, also taking into account that $x > 1$, we find that the resonant production takes place in the domain

$$x > x_- = \frac{2\zeta_n^2 - 1}{2(1 - y)} \left[\sqrt{1 + \frac{4\zeta_n^4(1 - y)}{(2\zeta_n^2 - 1)^2 y}} - 1 \right]. \quad (4.44)$$

It is worth noting that, for any mode, x_+ in (4.43) diverges as $y \rightarrow 1$ while the lower boundary x_- in (4.44) blows up as $y \rightarrow 0$ (i.e., as $\Lambda \rightarrow \Lambda_{\text{crit}}$, as expected). Note that the lower boundary in (4.43) and (4.44) is, by expression, the same function of y in different domains, so is denoted by the same symbol x_- .

The lowest mode ($\zeta_n^2 = 8$) is of particular interest because it determines the condition that no gravitons are resonantly produced. This mode is excited for the range of x determined

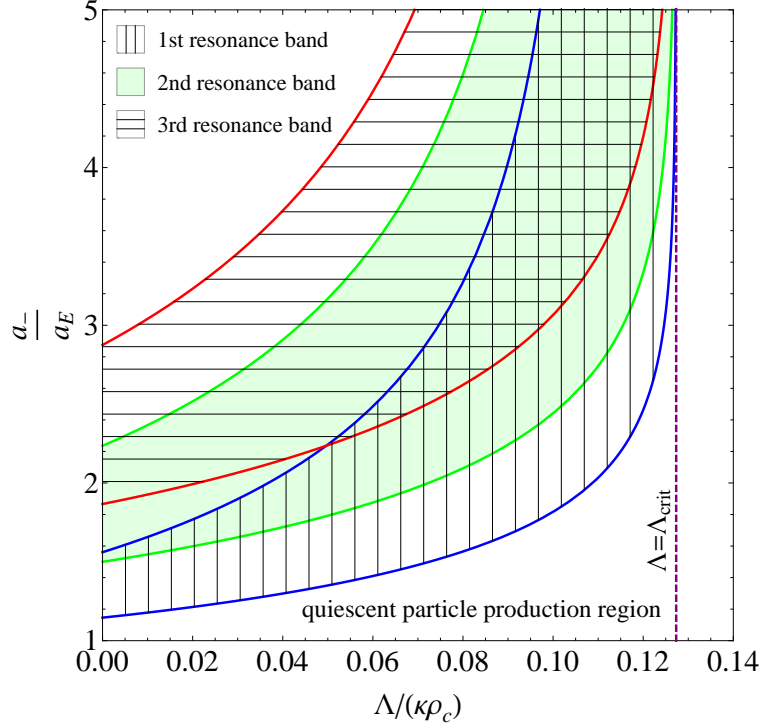


Figure 15. The resonance bands of the quantity a_-/a_E for three lowest graviton modes are shown as a function of the cosmological constant Λ . Again, the natural unit of $\kappa = 1$ is assumed along with $\rho_c = 1$. The “quiescent particle production region” below the first resonance domain corresponds to the region of parameters in which no graviton mode is in resonance.

by

$$x_{\pm} = \frac{15}{2(y-1)} \left[1 \pm \sqrt{1 - \frac{256(y-1)}{225y}} \right], \quad y > 1, \quad (4.45)$$

$$x_- = \frac{15}{2(1-y)} \left[\sqrt{1 + \frac{256(1-y)}{225y}} - 1 \right], \quad 0 < y < 1. \quad (4.46)$$

The resonance bands of the quantity a_-/a_E for three lowest modes are plotted as a function of Λ in Fig. 15. The values of y in (4.43) and (4.44) for different values of Λ are calculated from (3.20), (3.21) and (4.37). The overlapping bands (spanning all values of $\Lambda < \Lambda_{\text{crit}}$) again imply that, once a particular graviton mode is in the resonance, all the subsequent modes eventually will be resonantly excited. However, the range of a_-/a_E which does not excite any resonance mode actually expands with increasing Λ . This region, below the first resonance band, is indicated as the “quiescent particle production region”.

4.2 Non-resonant production of gravitons

In this section, we investigate more thoroughly the case where no resonant production of gravitons takes place, i.e., where no frequency ζ_n lies in the resonance band determined by condition (4.23). Although the graviton modes are not excited *resonantly*, their excitation

still takes place. Our aim is to show that this effect is rather small and does not destroy the stability of the oscillating universe.

Since all graviton modes are assumed to be out of the resonance band, we can apply perturbation theory for the calculation of their average occupation numbers. From the general theory of excitation of a harmonic oscillator with time-dependent frequency $\Omega(\eta)$, it is known that these occupation numbers are expressed through the complex Bogolyubov coefficients α and β . These coefficients, for boson fields, obey the normalization condition

$$|\alpha|^2 - |\beta|^2 = 1, \quad (4.47)$$

and satisfy the following system of equations (see, e.g., [33]):

$$\alpha' = \frac{\Omega'}{2\Omega} e^{+2i \int \Omega d\eta} \beta, \quad \beta' = \frac{\Omega'}{2\Omega} e^{-2i \int \Omega d\eta} \alpha. \quad (4.48)$$

If initially (for convenience, we set $\eta_0 = 0$) the oscillator is not excited, then one can set

$$\alpha(0) = 1, \quad \beta(0) = 0. \quad (4.49)$$

The average excitation number (average number of particles) at the time η is then given by

$$N(\eta) = |\beta(\eta)|^2. \quad (4.50)$$

In the case of graviton production, from (4.6), (4.11b) and (4.13), the time-dependent frequency of the corresponding harmonic oscillator can be identified as

$$\Omega_n^2 = \zeta_n^2 + \epsilon \zeta^2 \cos \zeta \eta. \quad (4.51)$$

For sufficiently small value of ϵ and outside the resonance band, we have

$$\frac{\Omega'_n}{2\Omega_n} \approx -\frac{1}{4} \frac{\epsilon \zeta^3 \sin \zeta \eta}{\zeta_n^2}. \quad (4.52)$$

To the first order in ϵ , from (4.48) and (4.49) we then have the following equation for β :

$$\beta'_n(\eta) \approx -\frac{1}{4} \frac{\epsilon \zeta^3 \sin \zeta \eta}{\zeta_n^2} e^{-2i\zeta_n \eta} = \frac{i\epsilon \zeta^3}{8\zeta_n^2} \left[e^{-i(2\zeta_n - \zeta)\eta} - e^{-i(2\zeta_n + \zeta)\eta} \right]. \quad (4.53)$$

Now integrating (4.53) from 0 to η , we have

$$\beta_n(\eta) \approx \frac{\epsilon \zeta^3}{8\zeta_n^2} \left[\frac{e^{-i(2\zeta_n + \zeta)\eta} - 1}{2\zeta_n + \zeta} - \frac{e^{-i(2\zeta_n - \zeta)\eta} - 1}{2\zeta_n - \zeta} \right]. \quad (4.54)$$

Hence,

$$\begin{aligned} |\beta_n(\eta)|^2 &\approx \left(\frac{\epsilon \zeta^3}{8\zeta_n^2} \right)^2 \left[\frac{e^{-i(2\zeta_n + \zeta)\eta} - 1}{2\zeta_n + \zeta} - \frac{e^{-i(2\zeta_n - \zeta)\eta} - 1}{2\zeta_n - \zeta} \right] \\ &\quad \times \left[\frac{e^{i(2\zeta_n + \zeta)\eta} - 1}{2\zeta_n + \zeta} - \frac{e^{i(2\zeta_n - \zeta)\eta} - 1}{2\zeta_n - \zeta} \right] \end{aligned} \quad (4.55)$$

or,

$$|\beta_n(\eta)|^2 \approx \left(\frac{\epsilon \zeta^3}{8 \zeta_n^2} \right)^2 \left(\frac{\sin^2(\zeta_n + \zeta/2) \eta}{(\zeta_n + \zeta/2)^2} + \frac{\sin^2(\zeta_n - \zeta/2) \eta}{(\zeta_n - \zeta/2)^2} + \frac{1}{2[\zeta_n^2 - (\zeta/2)^2]} [\cos(2\zeta_n + \zeta) \eta + \cos(2\zeta_n - \zeta) \eta - \cos 2\zeta \eta - 1] \right). \quad (4.56)$$

In the resonance, as $\zeta_n \rightarrow \zeta/2$, the second term in (4.54) dominates. When the resonance condition is not met, we can safely assume that $\zeta_n \gg \zeta/2$ (i.e., we are far away from the resonance). Equation (4.56) is then simplified as

$$|\beta_n(\eta)|^2 \approx \left(\frac{\epsilon \zeta^3}{8 \zeta_n^2} \right)^2 \frac{1}{\zeta_n^2} \left[2 \sin^2 \zeta_n \eta + \cos 2\zeta_n \eta - \frac{1}{2} (1 + \cos 2\zeta \eta) \right] = \left(\frac{\epsilon \zeta^3}{8} \right)^2 \frac{\sin^2 \zeta \eta}{\zeta_n^6}. \quad (4.57)$$

Now, to calculate the graviton energy density ρ_{grav} , the quantity $|\beta_n(\eta)|^2$ should be summed over the graviton modes with the energy of graviton taken into account, and then divided by the volume of the space, which is a_-^3 . We recall that the graviton modes on a three-sphere are labeled by the quantum numbers (4.4). The frequency depends only on n . Summing over l and m , we get

$$\sum_{l=2}^{n-1} \sum_{m=-l}^l = \sum_{l=2}^{n-1} (2l+1) = n^2 - 4. \quad (4.58)$$

Then, taking into account two polarizations and the fact that the energy of a graviton is ζ_n/a_- , we have

$$\begin{aligned} \rho_{\text{grav}} &= \frac{2}{a_-^4} \left(\frac{\epsilon \zeta^3}{8} \right)^2 \sin^2 \zeta \eta \sum_{n=3}^{\infty} \frac{n^2 - 4}{\zeta_n^5}, \\ &= \frac{2}{a_-^4} \left(\frac{\epsilon \zeta^3}{8} \right)^2 \sin^2 \zeta \eta \sum_{n=3}^{\infty} \frac{n^2 - 4}{(n^2 - 1)^{5/2}}, \\ &= \frac{2}{a_-^4} \left(\frac{\epsilon \zeta^3}{8} \right)^2 \sin^2 \omega t \sum_{n=3}^{\infty} \frac{n^2 - 4}{(n^2 - 1)^{5/2}}. \end{aligned} \quad (4.59)$$

The sum in this expression is convergent, and can be estimated as

$$\sum_{n=3}^{\infty} \frac{n^2 - 4}{(n^2 - 1)^{5/2}} \approx 0.06. \quad (4.60)$$

The graviton energy density turns out to be periodic with time as indicated in (4.59), with period equal to half the period of oscillation of the universe. The energy density of produced gravitons is shown as a function of conformal time η in Fig. 16 for a typical value of $\zeta = \sqrt{20}$, i.e., for the case of stiff-matter dominated universe, with no cosmological constant, oscillating very close to the ESU (see (4.33) and (4.38)).

From Fig. 14 and in the view of (4.38), it is apparent that, for small values of Λ , the quantity $\zeta/2$ cannot be regarded as much smaller than the frequency $\zeta_3 = \sqrt{8}$ of the first mode, which has the leading contribution in (4.59). Thus, when the ‘far away from resonance’

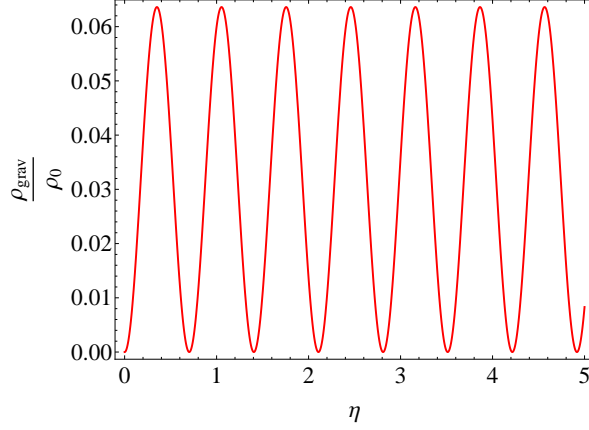


Figure 16. The energy density ρ_{grav} as a function of conformal time η in the case of graviton production far away from the resonance, when $\zeta_n \gg \zeta_{\text{res}} = \zeta/2$. Here, we choose $\zeta^2 = 20$ along with $\rho_0 = \frac{2}{a_-^4} \left(\frac{\epsilon \zeta^3}{8} \right)^2$.

assumption is not strictly valid, a more general expression for ρ_{grav} can be calculated directly from (4.56):

$$\rho_{\text{grav}} = \frac{2}{a_-^4} \left(\frac{\epsilon \zeta^3}{8} \right)^2 \sum_{n=3}^{\infty} \frac{n^2 - 4}{\zeta_n^3} \left(\frac{\sin^2(\zeta_n + \zeta/2)\eta}{(\zeta_n + \zeta/2)^2} + \frac{\sin^2(\zeta_n - \zeta/2)\eta}{(\zeta_n - \zeta/2)^2} + \frac{1}{2(\zeta_n^2 - (\zeta/2)^2)} [\cos(2\zeta_n + \zeta)\eta + \cos(2\zeta_n - \zeta)\eta - \cos 2\zeta\eta - 1] \right). \quad (4.61)$$

In Figs. 17(a) and 17(b), the quantity ρ_{grav} given by (4.61) is shown as a function of the conformal time η for $\zeta^2 = 6$ and $\zeta^2 = 20$, which can represent a radiation and stiff-matter dominated universe, respectively, with $\Lambda = 0$, while oscillating very close to the ESU. It is normalized by the quantity

$$\rho_0 = \frac{1}{32} \epsilon^2 \zeta^2 \omega^4 = \frac{1}{32} \left(\frac{\epsilon^2 \zeta_E^2}{1 - \epsilon^2 \zeta_E^2} \right) \omega^4 = \frac{1}{32 a_E^4} \left(\frac{\epsilon^2 \zeta_E^6}{1 - \epsilon^2 \zeta_E^2} \right), \quad (4.62)$$

where we have used the relation (see (4.38) and (4.39)),

$$\epsilon^2 \zeta^2 = \frac{\epsilon^2 \zeta_E^2}{1 - \epsilon^2 \zeta_E^2}.$$

It is worth noting that, in the limit $\epsilon^2 \rightarrow 1/\zeta_E^2$, the quantity ρ_0 (hence, also ρ_{grav}) becomes infinite. This is consistent with (4.59) and (4.61) since, in this limit, a_- (hence, also ζ) is also infinite (see (4.39)).

Now we have to compare the amount of graviton produced to the existing matter density. Although Figs. 17(a) and 17(b) show some departure from Fig. 16, we see that, in all cases, the energy density of the gravitons is small. Indeed, let us denote the summation in (4.61) by s (in (4.59), it is the summation multiplied by the time-dependent part). From (4.25), (3.13)

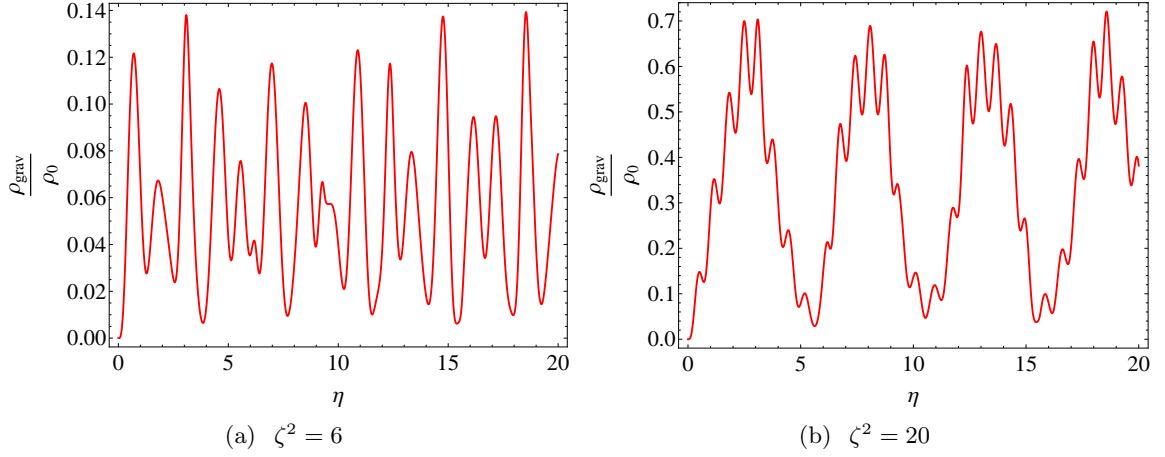


Figure 17. The energy density ρ_{grav} as a function of conformal time η with (a) $\zeta^2 = 6$ and (b) $\zeta^2 = 20$ in the case of non-resonant graviton production. These two cases represent a universe filled with radiation and stiff matter, respectively, with $\Lambda = 0$ and oscillating very closely to the ESU.

Again, $\rho_0 = \frac{2}{a_-^4} \left(\frac{\epsilon \zeta^3}{8} \right)^2$.

and (3.20), it is clear that ρ_c is a good estimation for the existing matter density. Hence, the ratio of the energy density of the produced gravitons to that of existing matter is

$$\frac{\rho_{\text{grav}}}{\rho_c} \approx 2s \left(\frac{\epsilon \zeta^3}{8} \right)^2 \frac{1}{a_-^4 \rho_c} = 2s \left(\frac{\epsilon \zeta}{8} \right)^2 \frac{\zeta_E^4}{a_E^4 \rho_c} = \frac{s \omega^4}{32 \rho_c} \left(\frac{\epsilon^2 \zeta_E^2}{1 - \epsilon^2 \zeta_E^2} \right) = \frac{s}{32 a_E^4 \rho_c} \left(\frac{\epsilon^2 \zeta_E^6}{1 - \epsilon^2 \zeta_E^2} \right). \quad (4.63)$$

In the last step, we used the general expression for ζ^2 from (4.38). Now we can see that $\frac{\rho_{\text{grav}}}{\rho_c} \ll 1$ due to the presence of the small parameter ϵ^2 on the right-hand side (all other quantities have finite values). For a typical case of stiff-matter dominated universe with $\Lambda = 0$, we can estimate the ratio using (3.17) and (4.33),

$$\frac{\rho_{\text{grav}}}{\rho_c} \approx \frac{s(\epsilon \zeta)^2}{a_E^2 M_p^2} = \frac{8}{5} \left(\frac{s \epsilon^2}{1 - 20 \epsilon^2} \right) \frac{\rho_c}{M_p^4} \ll 1. \quad (4.64)$$

Thus, for small deviations from the Einstein Static Universe, the maximum possible energy density of gravitons produced during the eternal oscillations in the emergent scenario appears to be tiny compared to the existing matter density, which ensures the stability of the model.

5 Conclusions

In this paper we have shown how the effective potential formalism can be used to study the dynamical properties of the emergent universe scenario. Within the GR setting, the effective potential has a single extreme point, a maximum, which corresponds to the *unstable* Einstein Static Universe (ESU). Extending our analysis to modified gravity theories we find that a new *minimum* in the effective potential appears corresponding to a *stable* ESU. These results are in broad agreement with earlier studies which also pointed out the appearance of an ESU in the context of extensions to GR [13, 14, 34]; also see [35].

While in GR, the emergent scenario can only occur if the universe is closed, we show that this restriction does not apply to certain modified gravity models in which the emergent scenario can occur in spatially closed as well as open cosmologies.

The appearance of a stable minimum in the effective potential considerably enlarges the initial data set from which the universe could have ‘emerged’. In this case, in addition to being precisely located at the minimum (ESU) – which requires considerable fine tuning of initial conditions, the universe can oscillate about it. Furthermore, we show that the existence of an ESU, while being conducive for emergent cosmology, is not essential for it. In section 3.3 this is demonstrated for LQC for which a stable ESU exists only for $\Lambda > \kappa\rho_c$ [14]. We demonstrate that even for $\Lambda \ll \kappa\rho_c$, when a stable ESU no longer exists, the universe can still oscillate about the minimum of its effective potential allowing an emergent scenario to be constructed.

However an oscillating universe is always accompanied by graviton production. While the magnitude of this semi-classical effect depends upon parameters in the effective potential, for a large region in parameter space this effect can be very large, casting doubts as to whether such an emergent scenario could have been past-eternal.⁵ (The instability of emergent cosmology to quantum effects has also been recently investigated in [36].) Although graviton production has been discussed in detail for an effective potential derived from the braneworld scenario, the effect itself is semi-classical and generic, and would be expected to accompany any emergent scenario in which the universe emerges from an oscillatory state.

One might also note that in the emergent scenarios discussed in this paper, the post-emergent universe inflates by well over 60 e-folds. Consequently any feature associated with the transition from an ESU to inflation is pushed to scales much larger than the present horizon. However it could well be that in some emergent scenarios this is not the case, and the transition from the ESU to inflation takes place fewer than ~ 60 e-folds from the end of inflation. In this case the spectrum of inflationary perturbations would differ from those considered in this paper on large scales, and may contain a feature in the CMB anisotropy spectrum, C_ℓ , at low values of ℓ .

Acknowledgments

V.S. and Yu.S. acknowledge support from the India-Ukraine Bilateral Scientific Cooperation programme. The work of Yu.S. was also partially supported by the SFFR of Ukraine Grant No. F53.2/028.

⁵ Graviton production is small, and does not stand in the way of emergent cosmology being past eternal, only if the universe oscillates very near the minimum of its effective potential. (Naturally, there is no particle production for a universe located precisely at the minimum of $U(a)$, i.e. for the ESU.) But this situation might require considerable fine tuning of parameters.

A Emergent scenario in a spatially open Braneworld

As mentioned in section 3.1, the braneworld admits a minimum in the effective potential $U(a)$ if $\Lambda > \kappa\rho_c$; see (3.8) and (3.9). Considering stiff matter along with $\Lambda > \kappa\rho_c$, from the equations (3.19) and (3.20), one finds that only (a_+, ρ_-) survive which now account for the *minimum* given by

$$a_+^6 = \frac{3A(2\Lambda - \kappa\rho_c)}{\Lambda(\Lambda - \kappa\rho_c)} \left[1 + \sqrt{1 + \frac{5\Lambda(\Lambda - \kappa\rho_c)}{(2\Lambda - \kappa\rho_c)^2}} \right], \quad (\text{A.1})$$

$$\rho_- = \rho_E = \frac{1}{5\kappa} (2\Lambda - \kappa\rho_c) \left[\sqrt{1 + \frac{5\Lambda(\Lambda - \kappa\rho_c)}{(2\Lambda - \kappa\rho_c)^2}} - 1 \right]. \quad (\text{A.2})$$

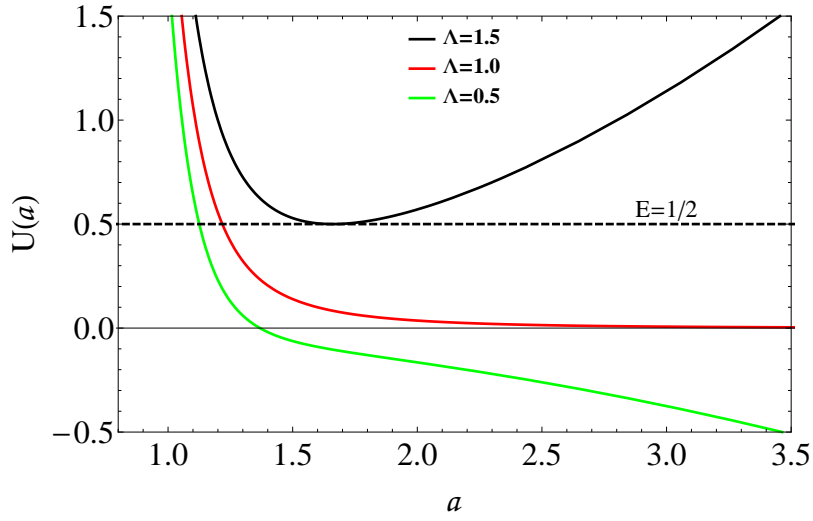


Figure 18. The effective potential for the braneworld in (3.6) is plotted for large values of Λ (in units $\kappa = 1$, $\rho_c = 1$). For $\Lambda > \kappa\rho_c$, a minima in $U(a)$ appears which can give rise to an ESU provided the universe is open ($\mathbb{k} = -1$). For $\Lambda \leq \kappa\rho_c$ the minimum disappears which allows the universe to exit the ESU and also to inflate (for a suitable choice of the inflaton potential).

According to (3.5), when $\Lambda > \kappa\rho_c$, a static solution can exist only for a spatially open universe ($\mathbb{k} = -1$). Again ρ_- is the energy density at the ESU, hence denoted by ρ_E in (A.2). The scale factor at ESU is calculated for the spatially open universe using (3.5) and (A.2) to be

$$a_E^{-2} = \frac{(\kappa\rho_E + \Lambda)}{3} \left(\frac{\rho_E}{\rho_c} + \frac{\Lambda}{\kappa\rho_c} - 1 \right). \quad (\text{A.3})$$

The effective potential in this case is plotted for various Λ in Fig. 18 which illustrates the emergence scenario in a spatially open universe. For $\Lambda > \kappa\rho_c$ the universe can either be an ESU or can oscillate around the minimum of $U(a)$ at a_+ . When $\Lambda = \kappa\rho_c$, the minimum disappears and, for large values of a , $U(a)$ asymptotically approaches zero from above. Therefore $\Lambda < \kappa\rho_c$ destabilizes the ESU and can result in inflation for the potential $V(\phi)$ in (2.16). After inflation commences the curvature term, \mathbb{k}/a^2 , rapidly declines to zero resulting in a spatially flat universe. (See [37] for a discussion of an emergent scenario in a spatially flat braneworld.)

B Linearization near a fixed point

A two dimensional non-linear system given by

$$\dot{x}_i = X_i(x_1, x_2) \quad \text{where } i = 1, 2, \quad (\text{B.1})$$

can be linearized in the neighbourhood of its simple fixed point (ζ, η) as [22],

$$\dot{x}_i \approx \mathcal{A}_{ij} x_j \quad \text{where} \quad \mathcal{A}_{ij} = \left. \frac{\partial X_i}{\partial x_j} \right|_{(\zeta, \eta)}. \quad (\text{B.2})$$

It should be noted that the *linearization theorem* guarantees that the linearized system in the neighbourhood of a fixed point is qualitatively equivalent to the non-linear system as long as the former does not suggest a centre type linearization.

For braneworld consisting of two component fluid: stiff matter with Λ , the linearized system is given by the co-efficient matrix in (B.2),

$$\mathcal{A} = \begin{bmatrix} 0 & -6\rho_{\pm} \\ -\frac{2}{3}\kappa \left\{ 1 - \frac{1}{\rho_c} \left(5\rho_{\pm} + \frac{2\Lambda}{\kappa} \right) \right\} & 0 \end{bmatrix}$$

The stability of fixed points of a linear system depends upon the eigenvalues of the Jacobian matrix \mathcal{A} (see chapter 2 of [22]). For a two dimensional system, if both eigenvalues are real and of opposite sign then the fixed point is a saddle in the phase portrait. On the other hand, if the eigenvalues are imaginary and complex conjugates of each other then the fixed point is likely to be a centre, but this must be confirmed numerically.

C CMB constraints on the Emergent Scenario in Braneworld Cosmology

The parameter γ in the potential (3.33) can be constrained using the CMB bounds on the scalar spectral index n_s and the tensor-to-scalar ratio r from the recent *Planck* mission [1]. As the scalar field rolls down the potential from $\phi \gg M$, the potential (3.33) is approximately the same as in the case of chaotic inflation models with $V(\phi) \propto \phi^{2\gamma}$. Therefore, in the slow roll limit one finds that

$$\begin{aligned} n_s - 1 &= -\frac{2(\gamma + 1)}{\gamma + 2N}, \\ n_T &= -\frac{2\gamma}{\gamma + 2N}, \\ r &= \frac{16\gamma}{\gamma + 2N}. \end{aligned} \quad (\text{C.1})$$

At 95% CL *Planck* data allows n_s within the range $[0.945 - 0.98]$. For n_s to lie in this allowed range with $N = 60$, the parameter γ must lie in the following range: $[0.202 - 2.26]$. Furthermore, *Planck* data also indicate that $r < 0.12$ at 95% CL when BAO data is included [1]. Using Eq. (C.1) and with $N = 60$, $r < 0.12$ can be realised only if $\gamma \leq 0.9$. Therefore, the potential (3.33) at $\phi \gg M$ can lead to both n_s within the range $[0.945 - 0.98]$ and $r < 0.12$ if the parameter γ is within the following range:

$$0.202 \leq \gamma \leq 0.9 \quad (\text{C.2})$$

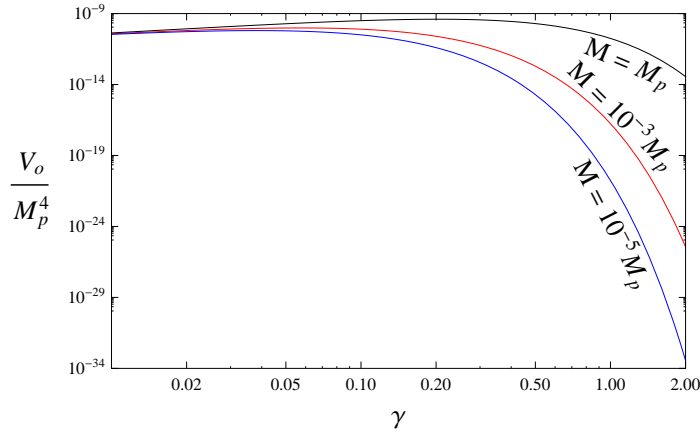


Figure 19. The CMB normalized value of V_0 in the potential (3.33) is plotted as a function of γ for three different values of M . In this figure we have taken the number of e-folds $N = 60$.

For example, if $\gamma = 0.8$, one gets $n_s \simeq 0.97$ and $r \simeq 0.1$ and these values are consistent with *Planck* results [1]. Note that in the potential (3.33), we have assumed that $\phi \gg M$ during inflation. Inflation ends at $\phi = \sqrt{2}\gamma M_p$. Therefore, the approximation $\phi \gg M$ is reasonable if $M \ll M_p$.

The constant V_0 can be fixed using the CMB normalization which indicate that $P_s(k_*) = 2.2 \times 10^{-9}$ at the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$ [1]. The expression for V_0 in terms of $P_s(k_*)$ is given by

$$\frac{V_0}{M_p^4} = \left(\frac{48\pi^2\gamma^2 P_s(k_*)}{[2\gamma(\gamma + 2N)]^{\gamma+1}} \right) \left(\frac{M}{M_p} \right)^{2\gamma}. \quad (\text{C.3})$$

Using the above equation, it turns out that $V_0 = 8 \times 10^{-16} M_p^4$ for $\gamma = 0.8$ and $M = 10^{-3} M_p$. In Fig. 19, the CMB normalized value of V_0 is plotted as a function of γ . One might note that in deriving (C.1) we assumed $\rho \ll \rho_c$ during inflation, with ρ_c defined in Eq. (3.3). At 60 e-folds before the end of inflation, one finds $\rho \simeq V(\phi) = 3.45 \times 10^{-9} M_p^4$ for $\gamma = 0.8$ irrespective of the value of M . Therefore, the approximation $\rho \ll \rho_c$ during inflation is valid provided $\rho_c \gg 10^{-9} M_p^4$.

D CMB constraints on the Emergent Scenario in LQC

As noted in section 3.3, the emergent scenario in LQC can proceed in two distinct ways:

- (i) If a minimum in the effective potential exists for *small* values of the inflaton potential. In this case the minimum gets destabilized as the inflaton potential *increases*, as shown in Fig. 13(a). A canonical scalar field potential such as (3.33) can accomplish this while satisfying CMB constraints.
- (ii) If a minimum in the effective potential exists for *large* values of the inflaton potential. In this case the minimum gets destabilized as the inflaton potential *decreases*, as shown in Fig. 13(b). While an inflaton potential such as (2.16) does accomplish this, it fails to satisfy CMB bounds if the scalar field has *canonical kinetic terms*. One therefore needs to turn to *non-canonical scalars* in order to construct a working example of the emergent scenario in this case, which forms the focus of this appendix.

Consider the following non-canonical scalar field Lagrangian [23]

$$\mathcal{L}(X, \phi) = X \left(\frac{X}{M^4} \right)^{\alpha-1} - V(\phi), \quad X = \frac{1}{2} \dot{\phi}^2, \quad (\text{D.1})$$

where α is a dimensionless parameter ($\alpha \geq 1$) while M has dimensions of mass. The canonical Lagrangian (2.12) corresponds to $\alpha = 1$ in (D.1). The energy density and pressure are modified for the non-canonical Lagrangian as follows:

$$\rho_\phi = (2\alpha - 1) X \left(\frac{X}{M^4} \right)^{\alpha-1} + V(\phi), \quad (\text{D.2a})$$

$$P_\phi = X \left(\frac{X}{M^4} \right)^{\alpha-1} - V(\phi). \quad (\text{D.2b})$$

The modified scalar field equation of motion is given by

$$\ddot{\phi} + \frac{3H\dot{\phi}}{2\alpha - 1} + \left(\frac{V'(\phi)}{\alpha(2\alpha - 1)} \right) \left(\frac{2M^4}{\dot{\phi}^2} \right)^{\alpha-1} = 0. \quad (\text{D.3})$$

As long as $V(\phi)$ is constant, the non-canonical scalar is equivalent to two non-interacting fluids: Λ (which mimics the constant potential V) plus matter with equation of state

$$w = \frac{1}{2\alpha - 1}. \quad (\text{D.4})$$

Thus the non-canonical formalism allows for a wider range of possibilities for the equation of state: $0 \leq w < 1$. For instance $\alpha = 2 \Rightarrow w = 1/3$, and the non-canonical scalar plays the role of a radiation+ Λ filled universe. The corresponding effective potential $U(a)$ is similar to that shown for the canonical scalar in Fig. 13(b), while the scale factor at ESU is determined from Table 2 to be $a_{LQ}^2 = 3(\Lambda - \kappa\rho_c)/2$.

We focus on the potential (2.16) within the non-canonical setting, assuming that during inflation $\lambda\phi \ll M_p$, so that (2.16) can be approximated as

$$V(\phi) \simeq V_0 \lambda^{2p} \left(\frac{\phi}{M_p} \right)^{2p}, \quad (\text{D.5})$$

which allows the problem to be tackled analytically.

Following [23] we find that in this case

$$\begin{aligned} n_s - 1 &= -2 \left(\frac{\sigma + p}{2N\sigma + p} \right), \\ n_T &= -\frac{2p}{2N\sigma + p}, \\ r &= \left(\frac{1}{\sqrt{2\alpha - 1}} \right) \left(\frac{16p}{2N\sigma + p} \right), \end{aligned} \quad (\text{D.6})$$

where

$$\sigma = \frac{\alpha + p(\alpha - 1)}{2\alpha - 1}. \quad (\text{D.7})$$

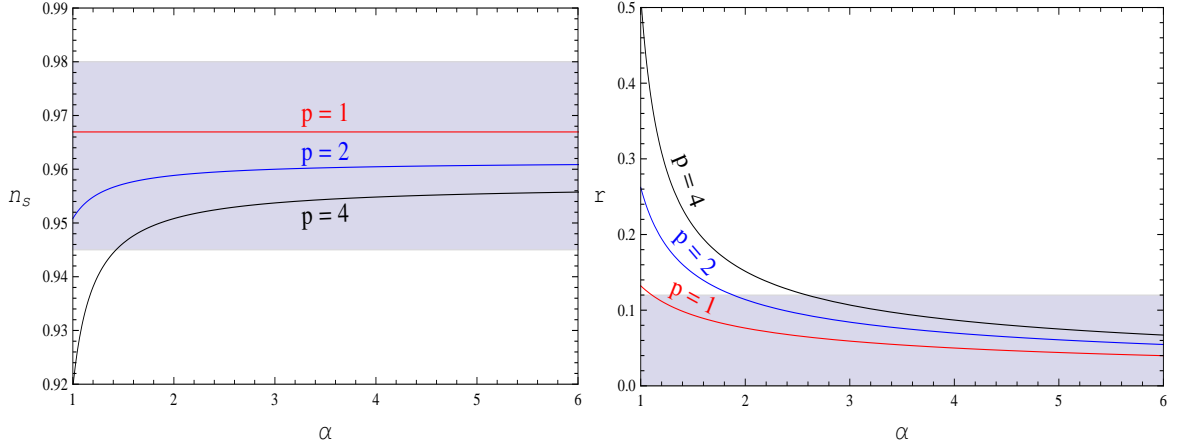


Figure 20. The scalar spectral index n_s (left panel) and the tensor-to-scalar ratio r (right panel) are shown as functions of α , for $p = 1$ and $p = 2$ in (2.16). The number of e-folds is fixed to $N = 60$. The shaded region refers to 95% confidence limits on n_s and r determined by Planck [1].

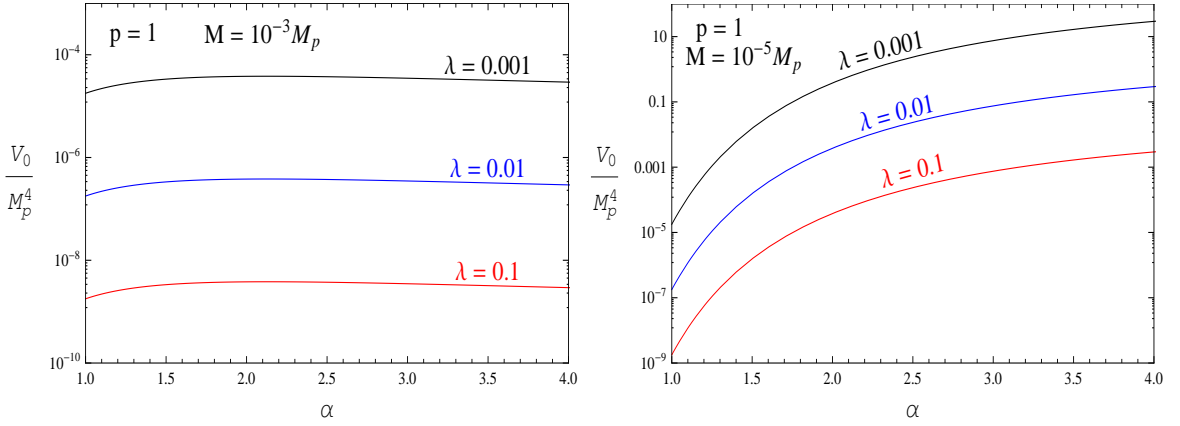


Figure 21. The CMB normalized value of V_0 in the potential (2.16) is plotted as a function of α . In the left panel, the value of the parameter M in (D.1) is set to be $10^{-3} M_p$, whereas $M = 10^{-5} M_p$ in the right panel.

In Fig. 20, the scalar spectral index n_s and the tensor-to-scalar ratio r are plotted as functions of α . Note that r decreases as α increases. Substituting $\alpha = 2$ in (D.6), we find

$$\begin{aligned} n_s &= 0.97 \quad \text{and} \quad r = 0.076 \quad \text{for} \quad p = 1, \\ n_s &= 0.96 \quad \text{and} \quad r = 0.11 \quad \text{for} \quad p = 2, \end{aligned} \quad (\text{D.8})$$

which satisfy the *Planck* requirements $n_s \in [0.945 - 0.98]$ and $r < 0.12$ at 95% CL [1].

The value of V_0 can be fixed using CMB normalization *viz.* $P_S(k_*) = 2.2 \times 10^{-9}$ at the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$ [1]. For (D.1) and (2.16), the expression for V_0 in terms of

$P_S(k_*)$ is given by

$$\frac{V_0}{M_p^4} = \left(\frac{1}{\lambda^{2p}} \right) \left\{ \left(\frac{24\pi^2 p P_S(k_*)}{\sqrt{2\alpha-1}} \right) \left[\left(\frac{\alpha}{2p} \right) \left(\frac{1}{6\mu^4} \right)^{\alpha-1} \right]^{\frac{p}{\sigma(2\alpha-1)}} \left(\frac{1}{2N\sigma+p} \right)^{\frac{\sigma+p}{\sigma}} \right\}^{\frac{\sigma(2\alpha-1)}{\alpha}}, \quad (\text{D.9})$$

where $\mu = M/M_p$. In Fig. 21 the CMB normalized value of V_0 is plotted as a function of α for different values of λ and M . One finds that when $M = 10^{-3}M_p$ the value of V_0 is nearly independent of α , whereas V_0 *increases dramatically* with increasing α , when $M < 10^{-3}M_p$. This result also holds when $p > 1$. Furthermore, for fixed values of α and M , V_0 increases as λ decreases, as shown in Fig. 22. From these figures it's clear that one can have $V_0 \gtrsim M_p$ by appropriately choosing the model parameters α , λ and M . For instance, $\alpha = 2$ and $M = 10^{-5}M_p$ result in

$$\begin{aligned} V_0 &\gtrsim M_p^4 \quad \text{when} \quad \lambda \lesssim 6.2 \times 10^{-4} \quad \text{for} \quad p = 1, \\ V_0 &\gtrsim M_p^4 \quad \text{when} \quad \lambda \lesssim 7.2 \times 10^{-2} \quad \text{for} \quad p = 2, \end{aligned} \quad (\text{D.10})$$

together with the *Planck*-consistent values for n_s and r quoted in (D.8).

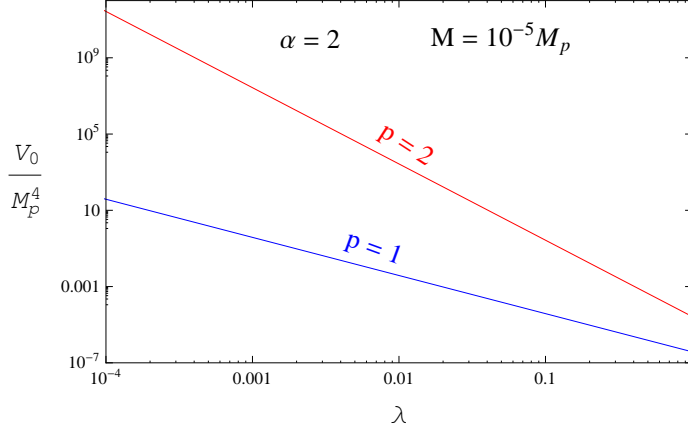


Figure 22. The CMB normalized value of V_0 in (2.16) is plotted as a function of λ . The two parameters α and M in the Lagrangian (D.1) have been fixed to $\alpha = 2$ and $M = 10^{-5}M_p$.

As mentioned earlier, these results are valid only when $\lambda\phi \ll M_p$ during inflation. In this case, N e-folds prior to the end of inflation, one finds

$$\phi(N) = C_1^{1/(2\sigma)} (2N\sigma + p)^{1/(2\sigma)} M_p, \quad (\text{D.11})$$

where σ was defined in (D.7) and C_1 is given by

$$C_1 = \left\{ \left(\frac{2p\mu^{4(\alpha-1)}}{\alpha} \right) \left(\frac{6M_p^4}{V_0\lambda^{2p}} \right) \right\}^{\frac{1}{2\alpha-1}}. \quad (\text{D.12})$$

For $\alpha = p = 2$, the above equations give $\phi = 0.1M_p$, 60 e-folds before the end of inflation, making the approximation $\lambda\phi \ll M_p$ perfectly reasonable provided $\lambda < 1$.

We therefore conclude that the emergent scenario in LQC based on the non-canonical model (D.1) with potential (2.16) can lead to CMB-consistent values for n_s and r even when $V_0 \gtrsim M_p^4$, provided that the semi-classical treatment followed by us is allowed in this regime.

References

- [1] P. A. R. Ade *et al.* [Planck Collaboration], *Planck 2013 results. XXII. Constraints on inflation*, arXiv:1303.5082 [astro-ph.CO].
- [2] A. Borde, A. H. Guth and A. Vilenkin, *Inflationary space-times are incomplete in past directions*, Phys. Rev. Lett. **90** (2003) 151301 [gr-qc/0110012];
A. Borde and A. Vilenkin, *Eternal inflation and the initial singularity*, Phys. Rev. Lett. **72** (1994) 3305 [gr-qc/9312022];
A. Borde and A. Vilenkin, *Violations of the weak energy condition in inflating space-times*, Phys. Rev. D **56** (1997) 717 [gr-qc/9702019].
- [3] A. S. Eddington, *On the Instability of Einstein's Spherical World*, Mon. Not. Roy. Astron. Soc. **90** (1930) 668.
- [4] G. Lemaitre, *A homogeneous Universe of constant mass and growing radius accounting for the radial velocity of extragalactic nebulae*, Annales Soc. Sci. Brux. Ser. I Sci. Math. Astron. Phys. A **47** (1927) 49;
G. Lemaitre, *A homogeneous universe of constant mass and increasing radius accounting for the radial velocity of extra-galactic nebulae*, Mon. Not. Roy. Astron. Soc. **91** (1931) 483.
- [5] A. Einstein, *Cosmological considerations in the General Theory of Relativity*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1917** (1917) 142.
- [6] E. Hubble, *A relation between distance and radial velocity among extra-galactic nebulae*, Proc. Nat. Acad. Sci. **15** (1929) 168.
- [7] A. Pais, *Subtle is the Lord. The science and the life of Albert Einstein*, Oxford University Press, (1982).
- [8] Ya. B. Zel'dovich, *The Cosmological constant and the theory of elementary particles*, Sov. Phys. Usp. **11** (1968) 381 [Usp. Fiz. Nauk **95** (1968) 209].
- [9] G. F. R. Ellis and R. Maartens, *The emergent universe: Inflationary cosmology with no singularity*, Class. Quant. Grav. **21** (2004) 223 [gr-qc/0211082].
- [10] G. F. R. Ellis, J. Murugan and C. G. Tsagas, *The Emergent universe: An Explicit construction*, Class. Quant. Grav. **21** (2004) 233 [gr-qc/0307112].
- [11] Y. Shtanov and V. Sahni, *Bouncing brane worlds*, Phys. Lett. B **557** (2003) 1 [gr-qc/0208047].
- [12] A. Ashtekar, T. Pawłowski and P. Singh, *Quantum nature of the big bang: An analytical and numerical investigation*, Phys. Rev. D **73** (2006) 124038 [gr-qc/0604013].
- [13] D. J. Mulryne, R. Tavakol, J. E. Lidsey and G. F. R. Ellis, *An emergent universe from a loop*, Phys. Rev. D **71** (2005) 123512 [astro-ph/0502589].
- [14] L. Parisi, M. Bruni, R. Maartens and K. Vandersloot, *The Einstein static universe in Loop Quantum Cosmology*, Class. Quant. Grav. **24** (2007) 6243 [arXiv:0706.4431 [gr-qc]].
- [15] V. Sahni and A. A. Starobinsky, *The Case for a positive cosmological Lambda term*, Int. J. Mod. Phys. D **9** (2000) 373 [astro-ph/9904398].
- [16] E. R. Harrison, *Classification of uniform cosmological models*, Mon. Not. Roy. Astron. Soc. **137** (1967) 69.
- [17] P. Labrana, *Emergent Universe Scenario and the Low CMB Multipoles*, arXiv:1312.6877 [astro-ph.CO].
- [18] A. A. Starobinsky, *A new type of isotropic cosmological models without singularity*, Phys. Lett. B **91** (1980) 99.
- [19] K. Maeda, *Inflation as a transient attractor in R^{*2} cosmology*, Phys. Rev. D **37** (1988) 858.

- [20] Y. Cai, Y. Wan and X. Zhang, *Cosmology of the spinor emergent universe and scale-invariant perturbations*, Phys. Lett. B **731** (2014) 217 [arXiv:1312.0740 [hep-th]];
Y. Cai, M. Li and X. Zhang, *Emergent universe scenario via Quintom matter*, Phys. Lett. B **718** (2012) 248 [arXiv:1209.3437 [hep-th]].
- [21] V. Sahni and A. Toporensky, *Cosmological hysteresis and the cyclic universe*, Phys. Rev. D **85** (2012) 123542 [arXiv:1203.0395 [gr-qc]].
- [22] D. K. Arrowsmith and C. K. Place, *Dynamical Systems: Differential Equations, Maps and Chaotic Behaviour*, Chapman and Hall: London, (1992).
- [23] S. Unnikrishnan, V. Sahni and A. Toporensky, *Refining inflation using non-canonical scalars*, JCAP **1208** (2012) 018 [arXiv:1205.0786 [astro-ph.CO]];
S. Unnikrishnan and V. Sahni, *Resurrecting power law inflation in the light of Planck results*, JCAP **1310** (2013) 063 [arXiv:1305.5260 [astro-ph.CO]];
S. Li and A. R. Liddle, *Observational constraints on K -inflation models*, JCAP **1210** (2012) 011 [arXiv:1204.6214 [astro-ph.CO]].
- [24] L. P. Grishchuk, *Amplification of gravitational waves in an isotropic universe*, Sov. Phys. JETP **40** (1975) 409 [Zh. Eksp. Teor. Fiz. **67** (1974) 825].
- [25] Y. Shtanov and A. Viznyuk, *Linearized gravity on the Randall-Sundrum two brane background with curvature terms in the action for the branes*, Class. Quant. Grav. **22** (2005) 987 [hep-th/0312261];
Y. Shtanov and A. Viznyuk, *Mirror branes*, Mod. Phys. Lett. A **20** (2005) 2283 [hep-th/0404077].
- [26] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press, (1982).
- [27] A. A. Grib, S. G. Mamaev and V. M. Mostepanenko, *Vacuum Effects in Strong Fields*, Friedmann Laboratory Publishing: St. Petersburg, (1994).
- [28] A. A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, JETP Lett. **30** (1979) 682 [Pisma Zh. Eksp. Teor. Fiz. **30** 719 (1979)].
- [29] B. Allen, *Stochastic gravity-wave background in inflationary-universe models*, Phys. Rev. D **37** (1988) 2078.
- [30] V. Sahni, *Energy density of relic gravity waves from inflation*, Phys. Rev. D **42** (1990) 453.
- [31] V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press, (2005).
- [32] B. Allen, *Maximally symmetric spin-two bitensors on S^{**3} and H^{**3}* , Phys. Rev. D **51** (1995) 5491 [gr-qc/9411023].
- [33] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, *Universe reheating after inflation*, Phys. Rev. D **51** (1995) 5438 [hep-ph/9407247].
- [34] C. G. Boehmer, L. Hollenstein and F. S. N. Lobo, *Stability of the Einstein static universe in $f(R)$ gravity*, Phys. Rev. D **76** (2007) 084005 [arXiv:0706.1663 [gr-qc]];
R. Goswami, N. Goheer and P. K. S. Dunsby, *Existence of Einstein static universes and their stability in fourth-order theories of gravity*, Phys. Rev. D **78** (2008) 044011 [arXiv:0804.3528 [gr-qc]];
N. Goheer, R. Goswami and P. K. S. Dunsby, *Dynamics of $f(R)$ -cosmologies containing Einstein static models*, Class. Quant. Grav. **26** (2009) 105003 [arXiv:0809.5247 [gr-qc]];
S. S. Seahra and C. G. Boehmer, *Einstein static universes are unstable in generic $f(R)$ models*, Phys. Rev. D **79** (2009) 064009 [arXiv:0901.0892 [gr-qc]];
S. Carneiro and R. Tavakol, *Stability of the Einstein static universe in presence of vacuum energy*, Phys. Rev. D **80** (2009) 043528 [arXiv:0907.4795 [astro-ph.CO]];
K. Zhang, P. Wu and H. W. Yu, *The Stability of Einstein static universe in the DGP braneworld*, Phys. Lett. B **690** (2010) 229 [arXiv:1005.4201 [gr-qc]].

- P. Labrana, *Emergent Universe by Tunneling*, Phys. Rev. D **86** (2012) 083524 [arXiv:1111.5360 [gr-qc]].
- S. del Campo, R. Herrera and P. Labrana, *Emergent universe in a Jordan-Brans-Dicke theory*, JCAP **0711** (2007) 030 [arXiv:0711.1559 [gr-qc]].
- S. del Campo, R. Herrera and P. Labrana, *On the stability of Jordan-Brans-Dicke static universe*, JCAP **0907** (2009) 006 [arXiv:0905.0614 [gr-qc]].
- [35] S. Mukherjee, B. C. Paul, N. K. Dadhich, S. D. Maharaj and A. Beesham, *Emergent Universe with Exotic Matter*, Class. Quant. Grav. **23** (2006) 6927 [gr-qc/0605134].
- [36] A. T. Mithani and A. Vilenkin, *Instability of an emergent universe*, arXiv:1403.0818 [hep-th].
- [37] K. Zhang, P. Wu and H. Yu, *Emergent universe in spatially flat cosmological model*, JCAP01(2014)048 [arXiv:1311.4051 [gr-qc]].